**TOPOLGY**

**Contents**

1. Topological space.

2. Special topologies on .

3. Subspace.

4. Limit points & Closure of sets.

5. Interior, Exterior & Boundary of sets.

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**Topological spaces**

**\* Topology and topological spaces:**

**Definition 1:** Let be a non-empty set. Then, a collection of subsets of is called a topology for , if satisfies the following axioms:-

1. ;
2. ;
3. If be an arbitrary collection of sets in , then

is in .

1. If and are any two sets in , then is in .

**Ex 1:** Let

The smallest topology on .

; The largest topology on .

.

Determine the topology on

Sol:

Is topology on .

Is topology on .

Is not topology on , since .

Is topology on .

Is not topology on , since .

**Note 1:**

1. Called the discrete topology on .
2. Called the indiscrete topology on .

**; The number of member of .**

**Definition 2:** (**Topological space) فضاء تبولوجي**

Let be a set and topology on . Then the pair is called a **topological** **space**.

**Definition 3: (Open sets) المجموعة المفتوحة**

If Topology on set , then each elements in are called **Open sets.**

**Definition 4: (Closed sets) المجموعة المغلقة**

The complements المتممات of elements of are called the **Closed sets.**

**Ex 2:** Let be a topology on , determine **Open sets, Closed sets, Not open & Not closed sets, Clopen sets.**

**Sol:**

1. **Open sets:**

.

1. **Closed sets:**

.

1. **Not open & Not closed sets:**

.

1. **Clopen sets:**

.

**Note 2:** **(Trivial topologies) تبولوجيات تافهة**

On the set are discrete & indiscrete topologies.

**H.W:**

1. Let , find 6 topologies on .
2. Let .

Find:

1. **Open sets.**
2. **Closed sets.**
3. **Not open & Not closed sets.**
4. **Clopen sets.**
5. **Open but not closed.**
6. **Closed but not open**
7. Construct a non-trivial topology in which closed sets are the same of open sets.
8. Let find every topologies on .

**Theorem 1:** Let be two topologies on . Then

1. is a topology on .
2. is not a topology on .

**Proof:**

a)

1. ;

.

1. Let

Is topology on

is topology on .

.

1. Let

To prove ??

;

So ,

Is topology on .

b) To show is not topology on , we give this example:

Let , then each is topology on .

But is not a topology on , since .

**Ex 3:** Let the set of all natural number , prove that is a topology on .

**Proof:**

1. .
2. Let

To prove

* If ;
* If ;
* If

1. Let

To prove

* If ;
* If ;
* If

Is topology on .

**\* Intersections & unions of closed sets**

**Note 2: (De-morgan’s Laws)**

1. ;
2. .

**Or**

**Theorem 2:**

1. Any intersections of closed sets is closed.
2. Finite unions of closed sets is closed.

**Proof:**

1. Let be any family of closed sets.

Now,

(by De-Morgan’s law)

is closed (given)

is open

1. Let be finite family of closed sets.

**Now,**

For each  **is open** (by 3 condition of topology)

**Special topologies on**

1. **The standard topology on (or usual topology)**

Let

Prove that is a topology on ? Where is called the standard topology.

1. **The left ray topology on** .

Let

**Prove that is topology on ?**

**Pf:**

1. (given)
2. Let where :
3. If .
4. If .
5. If .
6. Let :
7. If .
8. If .
9. If

.

1. **The right ray topology on** .

Let

**Prove that is topology on ?**

**Pf: H.W**

1. **The co-finite topology on** .

Let

**Prove that is topology on ?**

**Pf:**

**Ex 4**: Consider the following sets:

Determine which is Open, Closed, Clopen sets W. r. t. with respect to

**Sol**:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Sets |  |  |  |  |  |  |
|  | Clopen | Clopen | Clopen | Clopen | Clopen | Clopen |
|  | Clopen | Clopen | Clopen | Clopen | Clopen | Clopen |
|  | Open | Open | ------ | ------ | ------- | Clopen |
|  | ------ | ------- | ------- | ------- | ------ |  |
|  | Closed | Closed | ------ | ------ | ----- | Clopen |
|  | Open | ------ | ------ | ------ | ------ | Clopen |
|  | Closed | ------ | ------ | Closed | ------- | Clopen |
|  | Closed | ------- | ------ | Closed | ------- | Clopen |
|  | Open | ------- | Open | ------ | ------- | Clopen |
|  | Closed | ------ | ------- | ------ | ------ | Clopen |

**Subspace topology الفضاءات الجزئية**

**Theorem 3**: Let be a topological space & . Then the collection

Is a topology on called the subspace topology & is called subspace.

**Pf:**

1. Let be any family of .

Now,

1. Let

is a topology on .

**Ex 5**: If and

Find ?

**Sol:**

then

is a topology on .

**The closure of a set**

**Definition 5:** Let be a topological space , then the intersection of all closed supersets of is called the closure of and is denoted by . i.e.:

**Ex 6:** Let let . Find .

**Sol:**

1. .
2. .
3. .

**Theorem 5:**

1. is the smallest closed set contains .
2. closed .

**Proof:**

1. From definition of , we get
2. is the intersection of all closed set supersets of .

is the smallest.

1. is closed set
2. is closed (by definition)

.

.

1. The right side :

is closed (given)

is the smallest closed set contains itself.

(by (i) from 1)

The left side:

(given)

is closed (from 1)

is closed.

**\* Properties of closures of sets:**

**Theorem 6:** Let be a topological space & , then:

1. ;
2. ;
3. ;
4. ;
5. ;

**Cluster points (Limits points)**

**Definition 7:** Let be a topological space & , then is called a limit point and the set of all a limit point of is called the derived set & denoted by .

**Ex 7:** Let let

Find ?

**Sol:**

First consider . Then

.

Now, consider then ,

.

Similarly then

.

Consider then

.

.

**Ex 8:** Consider be a topological space, let . Find ?

**Sol:**

**Theorem 7:** Let , then .

**Theorem 8:** Let be a topological space & , then

1. closed .
2. .

**Proof:**

1. H.W

So .

Let

1. .
2. but by theorem 7

.

.

**Isolated points of a set النقاط المعزولة**

**Definition 8:** Let be a topological space & , then is called a isolated point and the set of all isolated point denoted by .

**Ex9:** Consider & find .

**Sol:**

If

If

.

**Ex 10:** Let let

Find ?

Sol:

**Interior, Exterior & Boundary of a set**

**Definition 9:** Let be a topological space & . Then

1. .
2. .

Or .

**Ex 11:** Let let

Find ?

**Sol:**

**Ex 12:** Consider & find ?

**Sol:**

**Properties of interior of a set**

**Theorem 9:** If be a topological space & . Then:

1. .
2. is the largest open set Contained .
3. .
4. .
5. .
6. .
7. .

**Proof:**

**Properties of exterior of set**

**Theorem 10:** If be a topological space & . Then:

1. .
2. .
3. .
4. .
5. is open set.
6. .
7. .
8. .
9. .
10. .
11. .

**Proof:**

**Properties of boundary of set**

**Theorem 11:** If be a topological space & . Then:

1. .
2. is a closed set.
3. .
4. .
5. .
6. .
7. .

**Proof:**

**Theorem 12:** If be a topological space & . Then

1. .
2. are disjoint.

**Proof:**

**Continuity**

**Definition 10:** Let be two topological spaces & be a function then we say that is continuous at a point iff .

**Theorem 13:** Let then the following equivalent:

1. is continuity.
2. .
3. is closed in , closed in .

**Proof:**

**Ex 13:** Let two topologies on respestively. Consider s.t. . Show that is continuity.

**Sol:**