Inner Product Spaces

An **inner product space** is a vector space with additional structure. More specifically, each pair of vectors is associated with a scalar quantity known as the **inner product** of the vectors. The inner product is denoted as $\langle ., . \rangle$.

The inner product generalizes the concept of a dot product in Euclidean space. An inner product is defined by the following axioms:

1. $\langle \mathbf{v} + \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

2. $\langle \alpha \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{v}, \mathbf{u} \rangle$

3. $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$

4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \iff \mathbf{v} = \mathbf{0}$

Axioms 1 and 2 are called the **linearity** properties. Linearity holds for the first argument of the inner product. This preserves the right-distributivity of multiplication that we see in a ring. That is, $(a + b)c = ac + bc$. We will prove that the linearity property also holds for the second argument.

Axiom 3 is called the **symmetric** property of the inner product. Intuitively, a product should not depend on the order of the operands. Note that for complex vector spaces, this axiom would be **conjugate symmetry**. That is,

$$\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$$

The conjugate is used for complex vector spaces. In real valued vector spaces, the conjugate can be ignored.

Axiom 4 is called the **positive definite** property. Intuitively, the product of a vector with itself should be greater than zero. Only the zero-vector multiplied by itself should result in the zero-vector.

Induced Norm

An inner product defined over a vector space induces a norm on the vector space. Thus, all inner product spaces are also normed vector spaces. This norm is,

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

We assert that $f(\mathbf{v}) = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ satisfies the axiom of a norm.

Proof:

Axiom 1 states that $\forall \mathbf{v} \in \mathcal{V}$ where $\mathcal{V}$ is a vector space, $||\mathbf{v}|| \geq 0$. We see that since $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\sqrt{c} \geq 0$. It follows that $f(\mathbf{v}) \geq 0$ and thus the first axiom holds.

Axiom 2 states that $||\alpha\mathbf{v}|| = |\alpha|\,||\mathbf{v}||$. We see that

$$\begin{aligned}
f(\alpha\mathbf{v}) &= \sqrt{\langle \alpha\mathbf{v}, \alpha\mathbf{v} \rangle} \\
&= \sqrt{\alpha^2 \langle \mathbf{v}, \mathbf{v} \rangle} \\
&= |\alpha| \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}
\end{aligned}$$

Thus, the second axiom holds.

Axiom 3 states $||\mathbf{v} + \mathbf{u}|| \leq ||\mathbf{v}|| + ||\mathbf{u}||$. We see that,

$$\begin{aligned}
f(\mathbf{u} + \mathbf{v}) &= \sqrt{\langle \mathbf{v} + \mathbf{u}, \mathbf{v} + \mathbf{u} \rangle} \\
&= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{u} \rangle} \\
&\leq \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{u} \rangle} \\
&\leq \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} + \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \\
&= f(\mathbf{v}) + f(\mathbf{u})
\end{aligned}$$

Thus, the third axiom holds. This establishes that $f(\mathbf{v})$ is a valid norm.

□

Properties

Left-linearity

Left-linearity also holds for inner products. We ignore complex-conjugates and assume we are not dealing with complex values. Then,

$$\langle \mathbf{v}, \mathbf{w} + \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle$$

and

$$\langle \mathbf{v}, \alpha\mathbf{w} \rangle = \alpha\langle \mathbf{v}, \mathbf{w} \rangle$$

Since linearity holds for both the right and left arguments, we see that the inner product is a linear mapping from the vector space to the field for which the scalars belong.

Proof:

First,

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} + \mathbf{u} \rangle &= \langle \mathbf{w} + \mathbf{u}, \mathbf{v} \rangle \\ &= \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle \\ &= \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle \end{aligned}$$

Second,

$$\begin{aligned} \langle \mathbf{v}, \alpha\mathbf{w} \rangle &= \langle \alpha\mathbf{w}, \mathbf{v} \rangle \\ &= \alpha\langle \mathbf{w}, \mathbf{v} \rangle \\ &= \alpha\langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

□

Inner products involving the zero-vector are zero

$\forall \mathbf{v} \in \mathcal{V}$ where $\mathcal{V}$ is a vector space,

$$\langle \mathbf{v}, \mathbf{0} \rangle = 0$$

and

$$\langle \mathbf{0}, \mathbf{v} \rangle = 0$$

Proof:

We need only prove $\langle \mathbf{v}, \mathbf{0} \rangle = 0$ and the second claim will follow from the symmetric property.

Assume for the sake of contradiction that $\langle \mathbf{v}, \mathbf{0} \rangle = a$ where $a \neq 0$. Then,

$$\alpha \langle \mathbf{v}, \mathbf{0} \rangle = \langle \mathbf{v}, \alpha \mathbf{0} \rangle$$

$$= \langle \mathbf{v}, \mathbf{0} \rangle$$

Thus, we reach the conclusion that

$$\alpha a = a$$

This is a contradiction if $a \neq 0$. Thus, it must be that $a = 0$

□

Orthogonality

Two vectors $\mathbf{v}$ and $\mathbf{w}$ in an inner product space are **orthogonal** if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. That is,

$$\mathbf{v} \perp \mathbf{w} \iff \langle \mathbf{v}, \mathbf{w} \rangle = 0$$

Orthogonality is an abstraction of the concept of two vectors being perpendicular. That is, if two vectors don't share any "components", they are deemed to be orthogonal. In a Euclidean vector space, orthogonal vectors lie at right-angles to each other.