1. Complex numbers

A *complex number* z is defined as an ordered pair

$$z = (x, y),$$

where x and y are a pair of real numbers. In usual notation, we write

$$z = x + iy,$$

where i is a symbol. The operations of addition and multiplication of complex numbers are defined in a meaningful manner, which force $i^2 = -1$. The set of all complex numbers is denoted by $\mathbb{C}$. Write

$$\text{Re } z = x, \quad \text{Im } z = y.$$

Since complex numbers are defined as ordered pairs, two complex numbers (x_1, y_1) and (x_2, y_2) are equal if and only if both their real parts and imaginary parts are equal. Symbolically,

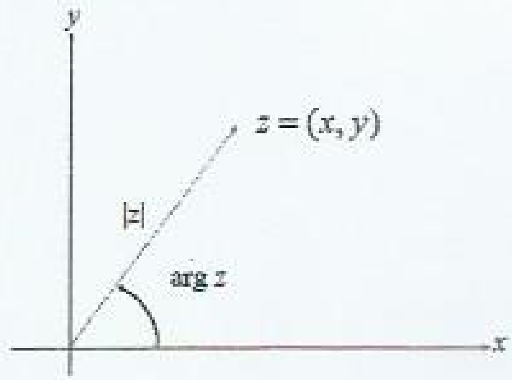
$$(x_1, y_1) = (x_2, y_2) \qquad \text{if and only if} \qquad x_1 = x_2 \quad \text{and} \quad y_1 = y_2 .$$

A complex number $z = (x, y)$, or as $z = x + iy$, is defined by a pair of real numbers x and y; so does for a point (x, y) in the x-y plane. We associate a one-to-one correspondence between the complex number $z = x + iy$ and the point (x, y) in the x-y plane. We refer the plane as the *complex plane* or z-plane.

Polar coordinates

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Modulus of z : $|z| = r = \sqrt{x^2 + y^2}$.



Vectorical representation of a complex number in the complex plane

Obviously, Re $z \leq |z|$ and Im $z \leq |z|$; and

$$z = x + iy = r(\cos\theta + i\sin\theta),$$

where θ is called the argument of z, denoted by arg z.

The principal value of arg z, denoted by Arg z, is the particular value of arg z chosen within the principal interval $(-\pi, \pi]$. We have

$$\text{arg } z = \text{Arg } z + 2k\pi \qquad k \text{ any integer}, \qquad \text{Arg } z \in (-\pi, \pi].$$

Note that arg z is a multi-valued function.

Complex conjugate

The complex conjugate $\overline{z}$ of $z = x + iy$ is defined by

$$\overline{z} = x - iy\,.$$

In the complex plane, the conjugate $\overline{z} = (x, -y)$ is the reflection of the point $z = (x, y)$ with respect to the real axis.

Standard results on conjugates and moduli

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, (ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, (iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$,

(iv) $|z_1 z_2| = |z_1|\,|z_2|$ (v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

also

The modulus has the following properties:

(a) $|\bar{z}| = |z|$;
(b) $z\bar{z} = |z|^2$;
(c) $|z_1 z_2| = |z_1||z_2|$, $|z_1/z_2| = |z_1|/|z_2|$;
(d) $|z_1 z_2 \ldots z_N| = |z_1||z_2|\ldots|z_N|$;
(e) $|z_1 + z_2| \leq |z_1| + |z_2|$, the triangle inequality;
(f) $|z_1 + z_2 + \cdots + z_N| \leq |z_1| + |z_2| + \cdots + |z_N|$;
(g) $|z_1 - z_2| \geq ||z_1| - |z_2||$.

Triangle inequalities

For any two complex numbers z_1 and z_2, we can establish

$$\begin{aligned} |z_1+z_2|^2 &= (z_1+z_2)(\overline{z_1}+\overline{z_2}) \\ &= z_1\overline{z_1}+z_2\overline{z_2}+z_1\overline{z_2}+z_2\overline{z_1} \\ &= |z_1|^2+|z_2|^2+2\mathrm{Re}(z_1\overline{z_2}). \end{aligned}$$

By observing that $\mathrm{Re}(z_1\overline{z_2}) \le |z_1\overline{z_2}|$, we have

$$\begin{aligned} |z_1+z_2|^2 &\le |z_1|^2+|z_2|^2+2|z_1\overline{z_2}| \\ &= |z_1|^2+|z_2|^2+2|z_1||z_2| = (|z_1|+|z_2|)^2. \end{aligned}$$

Since moduli are non-negative, we take the positive square root on both sides and obtain

$$|z_1+z_2| \le |z_1|+|z_2|.$$

To prove the other half of the triangle inequalities, we write

$$|z_1| = |(z_1+z_2)+(-z_2)| \le |z_1+z_2|+|-z_2|$$

giving

$$|z_1|-|z_2| \le |z_1+z_2|.$$

By interchanging z_1 and z_2 in the above inequality, we have

$$|z_2|-|z_1| \le |z_1+z_2|.$$

Combining all results together

$$\Big||z_1|-|z_2|\Big| \le |z_1+z_2| \le |z_1|+|z_2|.$$

Example
Evaluate i^{27} and i^{42}.

Addition or *subtraction* of complex numbers is accomplished by adding or subtracting the real and imaginary parts. Thus

$$(4+i5)+(3-i2)=4+i5+3-i2=(4+3)+i(5-2)=7+i3,$$

and

$$(4+i5)-(3-i2)=4+i5-3+i2=(4-3)+i(5+2)=1+i7,$$

In general, if $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, then

$$\begin{aligned} z_1+z_2 &= x_1+x_2+i(y_1+y_2), \\ z_1-z_2 &= x_1-x_2+i(y_1-y_2). \end{aligned}$$

Example
Evaluate $(5+i7)+(3-i4)-(6-i3)$.

Examples

1. Expand $(4-i5)^2$ and $(3+i4)(2-i5)(1-i2)$.
2. Find the possible values of the real numbers x and y such that

$$(x+iy)^2=i.$$

(In this example, we calculate the *square roots of i*.)

Examples

1. Find the complex conjugate of $(1+i2)(1-i3)$.
2. Show that for any two complex numbers z_1 and z_2,

$$\overline{z_1 z_2}=\overline{z_1}.\overline{z_2} \quad \text{and} \quad \overline{z_1+z_2}=\overline{z_1}+\overline{z_2}.$$

Examples

1. Express

$$\frac{2+i3}{i(4-i5)}-\frac{2}{i}$$

in the form $a+ib$.

2. If

$$\frac{1}{z} = \frac{1}{a} + \frac{1}{ib},$$

where a and b are real, find the real and imaginary parts of z, and show that

$$z\bar{z} = \frac{a^2 b^2}{a^2 + b^2}.$$

Examples

1. Find the modulus of (i) $(3 + i4)(5 - i12)$ and (ii) $(3 - i4)/(5 + i12)$.

2. If

$$\frac{1}{z} = \frac{1}{a} + \frac{1}{b + ic},$$

where a, b and c are real with $a > 0$, show that

$$|z| = a\sqrt{\frac{b^2 + c^2}{(a+b)^2 + c^2}}.$$

Examples

1. Plot the complex numbers $1+i$, $1-i$, $-1+i$, $-1-i$ on an Argand diagram. Find their moduli and arguments, expressing them in polar form, and verify that they lie on a circle in the Argand plane.

2. Given a complex number z, what points in the Argand plane correspond to

$$\bar{z}, \ -z, \ -\bar{z}?$$

3. Find the modulus and principal value of the argument of $z_1 = i$ and $z_2 = -1 - i\sqrt{3}$. Express these complex numbers in polar form. Find the value of

$$\arg(z_1 z_2) - \arg z_1 - \arg z_2,$$

where the arguments are principal values. What is the value of this combination of arguments if $z_1 = -i$?

Manipulation of complex numbers in polar form

Suppose that in polar form, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\
&= r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}.
\end{aligned}
$$

Thus

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2| \quad \text{and} \quad \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2,$$

Consider next the quotient z_1/z_2; then

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2} \\
&= \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}.
\end{aligned}
$$

Thus,

$$|z_1/z_2| = r_1/r_2 = |z_1|/|z_2| \quad \text{and} \quad \arg(z_1/z_2) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2,$$

De' Moivre's theorem

Any complex number with unit modulus can be expressed as $\cos\theta + i\sin\theta$. By virtue of the complex exponential function*, we have

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The above formula is called the *Euler formula*. As motivated by the Euler formula, one may deduce that

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta,$$

where n can be any positive integer.

* The complex exponential function is defined by

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y), \quad z = x + iy.$$

To prove the theorem, we consider the following cases:

(i) The theorem is trivial when $n = 0$.

(ii) When n is a positive integer, the theorem can be proven easily by mathematical induction.

(iii) When n is a negative integer, let $n = -m$ where m is a positive integer. We then have

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^n &= \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta} \\
&= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)} \\
&= \cos m\theta - i\sin m\theta = \cos n\theta + i\sin n\theta.
\end{aligned}
$$

How do we generalize the formula to $(\cos\theta + i\sin\theta)^s$, where s is a rational number?

Let $s = p/q$, where p and q are irreducible integers. Note that the modulus of $\cos\theta + i\sin\theta$ is one, so does $(\cos\theta + i\sin\theta)^s$. Hence, the polar representation of $(\cos\theta + i\sin\theta)^s$ takes the form $\cos\phi + i\sin\phi$ for some ϕ. Now, we write

$$\cos\phi + i\sin\phi = (\cos\theta + i\sin\theta)^s = (\cos\theta + i\sin\theta)^{p/q}.$$

Taking the power of q of both sides

$$\cos q\phi + i\sin q\phi = \cos p\theta + i\sin p\theta.$$

which implies

$$q\phi = p\theta + 2k\pi \quad \text{or} \quad \phi = \frac{p\theta + 2k\pi}{q}, \quad k = 0, 1, \cdots, q-1.$$

The value of ϕ corresponding to k that is beyond the above set of integers would be equal to one of those values defined in the equation plus some multiple of 2π.

There are q distinct roots of $(\cos\theta + i\sin\theta)^{p/q}$, namely,

$$\cos\left(\frac{p\theta + 2k\pi}{q}\right) + i\sin\left(\frac{p\theta + 2k\pi}{q}\right), \quad k = 0, 1, \cdots, q-1.$$

n^{th} root of unity

By definition, any n^{th} roots of unity satisfies the equation

$$z^n = 1.$$

By de' Moivre's theorem, the n distinct roots of unity are

$$z = e^{i2k\pi/n} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, ..., n-1.$$

If we write $\omega_n = e^{i2\pi/n}$, then the n roots are $1, \omega_n, \omega_n^2, ..., \omega_n^{n-1}$.

Alternatively, if we pick any one of the n^{th} roots and call it α, then the other $n-1$ roots are given by $\alpha\omega_n, \alpha\omega_n^2, ..., \alpha\omega_n^{n-1}$. This is obvious since each of these roots satisfies

$$(\alpha\omega_n^k)^n = \alpha^n(\omega_n^n)^k = 1, \quad k = 0, 1, \cdots, n-1.$$

In the complex plane, the n roots of unity correspond to the n vertices of a regular n-sided polygon inscribed inside the unit circle, with one vertex at the point $z = 1$. The vertices are equally spaced on the circumference of the circle. The successive vertices are obtained by increasing the argument by an equal amount of $2\pi/n$ of the preceding vertex.

Suppose the complex number in the polar form is represented by $r(\cos\phi + i\sin\phi)$, its n^{th} roots are given by

$$r^{1/n}\left(\cos\frac{\phi+2k\pi}{n} + i\sin\frac{\phi+2k\pi}{n}\right), \qquad k = 0, 1, 2, ..., n-1,$$

where $r^{1/n}$ is the real positive n^{th} root of the real positive number r. The roots are equally spaced along the circumference with one vertex being at $r^{1/n}[\cos(\phi/n) + i\sin(\phi/n)]$.

Examples

1. Find the three cube roots of (i) 1 and (ii) -1.
2. Find the two square roots of (i) i and (ii) $1 + i\sqrt{3}$.

Examples

1. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
2. $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.