Geometric applications

How to find the equation of the perpendicular bisector of the line segment joining the two points z_1 and z_2? Since any point z on the bisector will be equidistant from z_1 and z_2, the equation of the bisector can be represented by

$$|z - z_1| = |z - z_2|.$$

For a given equation $f(x, y) = 0$ of a geometric curve, if we set $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$, the equation can be expressed in terms of the pair of conjugate complex variables z and $\overline{z}$ as

$$f(x, y) = f\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right) = F(z, \overline{z}) = 0.$$

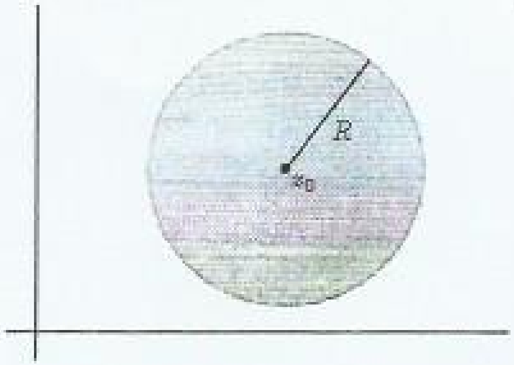
For example, the unit circle centered at the origin as represented by the equation $x^2 + y^2 = 1$ can be expressed as $z\overline{z} = 1$.

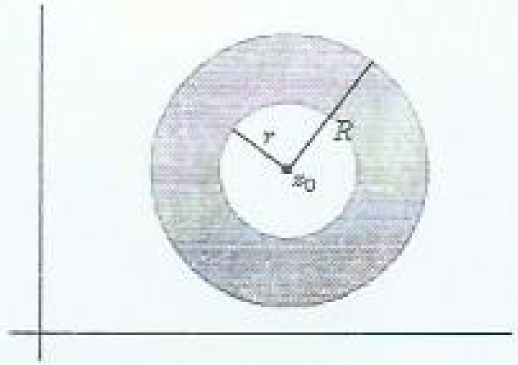
2.2 Point Sets in the Complex Plane

We shall study functions of the form $f : S \to \mathbb{C}$, where S is a set in $\mathbb{C}$. In most situations, various properties of the point sets S play a crucial role in our study. We therefore begin by discussing various types of point sets in the complex plane.

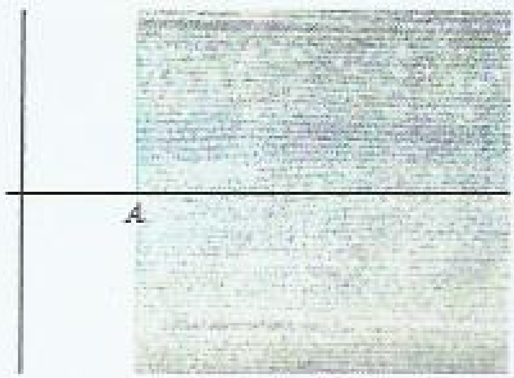
Before making any definitions, let us consider a few examples of sets which frequently occur in our subsequent discussion.

EXAMPLE 2.2.1. Suppose that $z_0 \in \mathbb{C}$, $r, R \in \mathbb{R}$ and $0 < r < R$. The set $\{z \in \mathbb{C} : |z - z_0| < R\}$ represents a disc, with centre z_0 and radius R, and the set $\{z \in \mathbb{C} : r < |z - z_0| < R\}$ represents an annulus, with centre z_0, inner radius r and outer radius R.





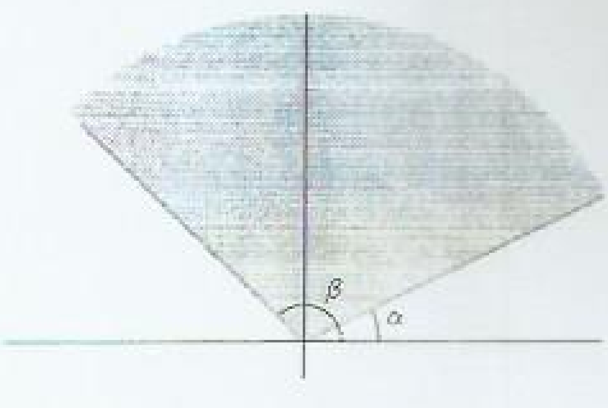
EXAMPLE 2.2.2. Suppose that $A, B \in \mathbb{R}$ and $A < B$. The set $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } x > A\}$ represents a half-plane, and the set $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } A < x < B\}$ represents a strip.



EXAMPLE 2.2.3. Suppose that $\alpha, \beta \in \mathbb{R}$ and $0 \le \alpha < \beta < 2\pi$. The set

$$\{z = r(\cos\theta + i\sin\theta) \in \mathbb{C} : r, \theta \in \mathbb{R} \text{ and } r > 0 \text{ and } \alpha < \theta < \beta\}$$

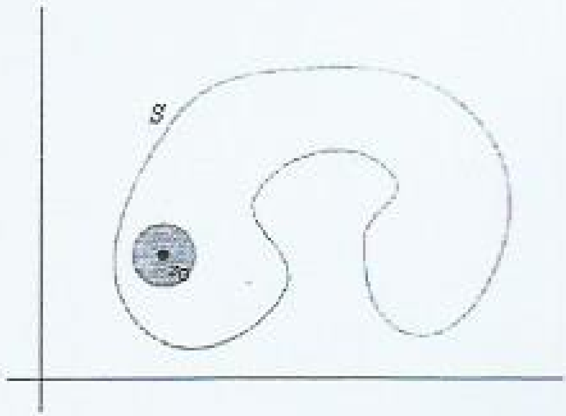
represents a sector.



We now make a number of important definitions. The reader may subsequently need to return to these definitions.

DEFINITION. Suppose that $z_0 \in \mathbb{C}$ and $\epsilon \in \mathbb{R}$, with $\epsilon > 0$. By an ϵ-neighbourhood of z_0, we mean a disc of the form $\{z \in \mathbb{C} : |z - z_0| < \epsilon\}$, with centre z_0 and radius $\epsilon > 0$.

DEFINITION. Suppose that S is a point set in $\mathbb{C}$. A point $z_0 \in S$ is said to be an interior point of S if there exists an ϵ-neighbourhood of z_0 which is contained in S. The set S is said to be open if, every point of S is an interior point of S.



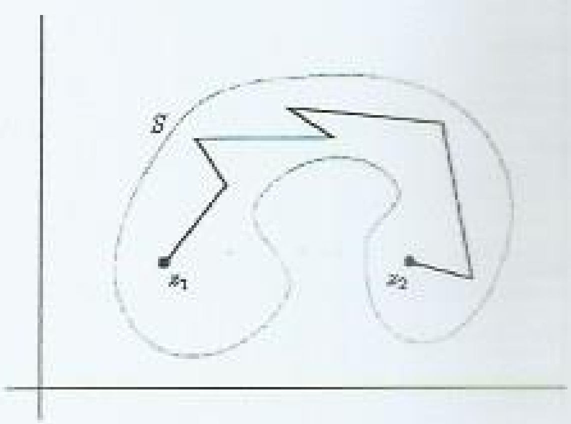
EXAMPLE 2.2.4. The sets in Examples 2.2.1–2.2.3 are open.

EXAMPLE 2.2.5. The punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ is open.

EXAMPLE 2.2.6. The disc $\{z \in \mathbb{C} : |z - z_0| \le R\}$ is not open.

EXAMPLE 2.2.7. The empty set $\emptyset$ is open. Why?

DEFINITION. An open set S is said to be connected if every two points $z_1, z_2 \in S$ can be joined by the union of a finite number of line segments lying in S. An open connected set is called a domain.



REMARKS. (1) Sometimes, we say that an open set S is connected if there do not exist non-empty open sets S_1 and S_2 such that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$. In other words, an open connected set cannot be the disjoint union of two non-empty open sets.

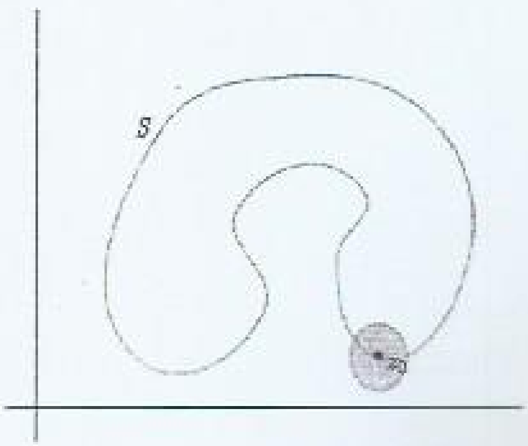
(2) In fact, it can be shown that the two definitions are equivalent.

(3) Note that we have not made any definition of connectedness for sets that are not open. In fact, the definition of connectedness for an open set given by (1) here is a special case of a much more complicated definition of connectedness which applies to all point sets.

EXAMPLE 2.2.8. The sets in Examples 2.2.1–2.2.3 are domains.

EXAMPLE 2.2.9. The punctured disc $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ is a domain.

DEFINITION. A point $z_0 \in \mathbb{C}$ is said to be a boundary point of a set S if every ϵ-neighbourhood of z_0 contains a point in S as well as a point not in S. The set of all boundary points of a set S is called the boundary of S.



EXAMPLE 2.2.10. The annulus $\{z \in \mathbb{C} : r < |z - z_0| < R\}$, where $0 < r < R$, has boundary $C_1 \cup C_2$, where

$$C_1 = \{z \in \mathbb{C} : |z - z_0| = r\} \qquad \text{and} \qquad C_2 = \{z \in \mathbb{C} : |z - z_0| = R\}$$

are circles, with centre z_0 and radius r and R respectively. Note that the annulus is connected and hence a domain. However, note that its boundary is made up of two separate pieces.

DEFINITION. A region is a domain together with all, some or none of its boundary points. A region which contains all its boundary points is said to be closed. For any region S, we denote by $\overline{S}$ the closed region containing S and all its boundary points, and call $\overline{S}$ the closure of S.

REMARK. Note that we have not made any definition of closedness for sets that are not regions. In fact, our definition of closedness for a region here is a special case of a much more complicated definition of closedness which applies to all point sets.

DEFINITION. A region S is said to be bounded or finite if there exists a real number M such that $|z| \leq M$ for every $z \in S$. A region that is closed and bounded is said to be compact.

EXAMPLE 2.2.11. The region $\{z \in \mathbb{C} : |z - z_0| \leq R\}$ is closed and bounded, hence compact. It is called the closed disc with centre z_0 and radius R.

EXAMPLE 2.2.12. The region $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } 0 \leq x \leq 1\}$ is closed but not bounded.

EXAMPLE 2.2.13. The square $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } 0 \leq x \leq 1 \text{ and } 0 < y < 1\}$ is bounded but not closed.

Example

Find the region in the complex plane that is represented by

$$0 < \text{Arg}\,\frac{z-1}{z+1} < \frac{\pi}{4}.$$

Solution

Let $z = x + iy$, and consider $\text{Arg}\,\dfrac{z-1}{z+1} = \text{Arg}\,\dfrac{x^2+y^2-1+2iy}{(x+1)^2+y^2}$, whose value lies between 0 and $\pi/4$ if and only if the following 3 conditions are satisfied

(i) $x^2+y^2-1>0$, (ii) $y>0$ and (iii) $\dfrac{2y}{x^2+y^2-1}<1$.

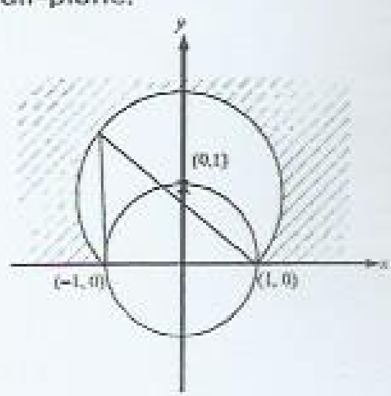
These 3 conditions correspond to

$$\text{Re}\,\frac{z-1}{z+1} > 0, \quad \text{Im}\,\frac{z-1}{z+1} > 0 \quad \text{and} \quad \text{Arg}\,\frac{z-1}{z+1} < \frac{\pi}{4}.$$

The last inequality can be expressed as $x^2+(y-1)^2>2$. For $y>0$, the region outside the circle: $x^2+(y-1)^2=2$ is contained completely inside the region outside the circle: $x^2+y^2=1$. Hence, the region represented by the above 3 inequalities is

$$R = \{x+iy : x^2+(y-1)^2>2 \quad \text{and} \quad y>0\}.$$

This is the region which is outside the circle $x^2+(y-1)^2=2$ and lying in the upper half-plane.



Example

Show that the boundary of

$$B_r(z_0) = \{z : |z - z_0| < r\}$$

is the circle: $|z - z_0| = r$.

Solution

Pick a point z_1 on the circle $|z - z_0| = r$. Every disk that is centered at z_1 will contain (infinitely many) points in $B_r(z_0)$ and (infinitely many) points not in $B_r(z_0)$. Hence, every point on the circle $|z - z_0| = r$ is a boundary point of $B_r(z_0)$.

No other points are boundary points

- Since points inside the circle are interior points, they cannot be boundary points.
- Given a point outside the circle, we can enclose it in a disk that does not intersect the disk $B_r(z_0)$. Hence, such a point is not boundary point.

In this example, none of the boundary points of $B_r(z_0)$ belong to $B_r(z_0)$.

Example

The set $R = \{z : \text{Re } z > 0\}$ is an open set. To see this, for any point $w_0 \in R$; then $\sigma_0 = \text{Re } w_0 > 0$. Let $\varepsilon = \sigma_0/2$ and suppose that $|z - w_0| < \varepsilon$. Then $-\varepsilon < \text{Re } (z - w_0) < \varepsilon$ and so

$$\text{Re } z = \text{Re } (z - w_0) + \text{Re } w_0 > -\varepsilon + \sigma_0 = \sigma_0/2 > 0.$$

Consequently, z also lies in R. Hence, each point w_0 of R is an interior point and so R is open.