27*

H.W 2

(تمارين محلولة)

1. $\int \frac{e^{x}\,dx}{\sqrt{4-e^{2x}}} = \int \frac{e^{x}\,dx}{\sqrt{4-(e^{x})^2}}$

sol. let $u = e^{x} \Rightarrow \frac{du}{dx} = e^{x} \Rightarrow du = e^{x}dx$

$\therefore \int \frac{e^{x}\,dx}{\sqrt{4-(e^{x})^2}} = \int \frac{du}{\sqrt{4-u^2}} = \sin^{-1}(\frac{u}{2}) + c = \sin^{-1}(\frac{e^{x}}{2}) + c$

2. $\int \frac{x\,dx}{\sqrt{4-3x^4}} = \int \frac{x\,dx}{\sqrt{4-(\sqrt{3}x^2)^2}} \Rightarrow$ let $u = \sqrt{3}x^2 \Rightarrow du = 2\sqrt{3}\,x\,dx$

$\Rightarrow \frac{du}{2\sqrt{3}} = x\,dx$

$\therefore \int \frac{x\,dx}{\sqrt{4-3x^4}} = \int \frac{\frac{du}{2\sqrt{3}}}{\sqrt{4-u^2}} = \frac{1}{2\sqrt{3}} \int \frac{du}{\sqrt{4-u^2}} = \frac{1}{2\sqrt{3}} \sin^{-1}(\frac{u}{2}) + c = \frac{1}{2\sqrt{3}} \sin^{-1}(\frac{\sqrt{3}x^2}{2}) + c$

3. $\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{3^2-(2x)^2}}$ let $u = 2x \Rightarrow du = 2\,dx \Rightarrow \frac{du}{2} = dx$

$\therefore \int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{\frac{du}{2}}{\sqrt{9-u^2}} = \frac{1}{2}\int \frac{du}{\sqrt{3^2-u^2}} = \frac{1}{2}\sin^{-1}(\frac{u}{3}) + c = \frac{1}{2}\sin^{-1}(\frac{2x}{3}) + c$

4. $\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}} = \int \frac{e^{-x}\,dx}{\sqrt{1-(e^{-x})^2}} \Rightarrow$ let $u = e^{-x} \Rightarrow du = -e^{-x}dx \Rightarrow -du = e^{-x}dx$

$\therefore \int \frac{e^{-x}\,dx}{\sqrt{1-e^{-2x}}} = \int \frac{-du}{\sqrt{1-u^2}} = -\int \frac{du}{\sqrt{1-u^2}} = -\sin^{-1}(u) + c = -\sin^{-1}(e^{-x}) + c$

5. $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$ let $u = \ln x \Rightarrow du = \frac{dx}{x} \Rightarrow dx = x\,du$

$\therefore \int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + c = \sin^{-1}(\ln x) + c$

6. $\int \frac{x\,dx}{1+x^4} = \int \frac{x\,dx}{1+(x^2)^2}$ let $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{du}{2} = x\,dx$

$\therefore \int \frac{x\,dx}{1+x^4} = \int \frac{\frac{du}{2}}{1+u^2} = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}(u) + c = \frac{1}{2}\tan^{-1}(x^2) + c$

7. $\int \frac{dx}{\sqrt{x}\,(1+x)}$ let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{dx}{\sqrt{x}}$

or $u^2 = x$

$\therefore \int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{dx}{\sqrt{x}} \cdot \frac{1}{(1+u^2)} = \int 2\,du \cdot \frac{1}{1+u^2} = 2\int \frac{du}{1+u^2} = 2\tan^{-1}(u) + c$

$= 2\tan^{-1}(\sqrt{x}) + c$

طرق التكامل

أ- التكامل بالتعويض : Integration by substitution

اذا كانت $f(x)$ و $g'(x)$ دالتان مستمرتان فأن التكامل

$$\int f(g(x))\, g'(x)\, dx$$

يمكن حله بوضع $u = g(x)$ وحساب $\int f(u)\, du$

وأن تحويل التكامل $\int f(g(x))\, g'(x)\, dx$

الى تكامل $\int f(u)\, du$ يتم من خلال تعويض

$$u = g(x) \quad , \quad du = g'(x)\, dx$$

وتسمى هذه الطريقة بطريقة التكامل بالتعويض وايضاً

تستعمل للتخلص من الجذور وتسهيلها الى صيغة أبسط.

* أمثلة :

1. $\displaystyle\int \frac{x\, dx}{\sqrt{3-x^2}}$

Sol. Let $u = 3 - x^2 \Rightarrow du = -2x\, dx \Rightarrow \boxed{\frac{1}{-2}\, du = x\, dx}$

الان نعوض في التكامل المعطى

$$\therefore \int \frac{x\, dx}{\sqrt{3-x^2}} = \int -\frac{1}{2} \frac{du}{\sqrt{u}} = -\frac{1}{2} \int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{2} \int \frac{du}{u^{\frac{1}{2}}} = -\frac{1}{2} \int u^{-\frac{1}{2}}\, du$$

$$= -\frac{1}{\cancel{2}} \frac{u^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} + c = -u^{\frac{1}{2}} + c = -\sqrt{u} + c$$

$$= -\sqrt{3-x^2} + c$$

2. $\int \frac{1}{(x^2+6x+9)}\, dx$

Sol.

عليه نكتب التكامل اعلاه بالشكل

$$\int \frac{1}{x^2+6x+9}\, dx = \int \frac{1}{(x+3)^2}\, dx$$

الآن نفرض

Let $u = x+3 \Rightarrow du = dx$

$$\therefore \int \frac{1}{(x+3)^2}\, dx = \int \frac{1}{u^2}\, du = \int u^{-2}\, du = \frac{u^{-1}}{-1} + c$$

$$= -\frac{1}{u} + c = -\frac{1}{x+3} + c$$

3. $\int x^3 \cos(x^4+1)\, dx$

Sol. Let $u = x^4+1 \Rightarrow du = 4x^3\, dx \Rightarrow \frac{1}{4} du = x^3 dx$

$$\therefore \int x^3 \cos(x^4+1)\, dx = \int \cos u \cdot \frac{1}{4}\, du = \frac{1}{4}\int \cos u\, du$$

$$= \frac{1}{4}\sin u + c = \frac{1}{4}\sin(x^4+1) + c$$

4. $\int x\sqrt{x+1}\, dx$

Sol. Let $u = \sqrt{x+1} \Rightarrow u^2 = x+1 \Rightarrow 2u\, du = dx$

$\Rightarrow x = u^2 - 1$

$$\therefore \int x\sqrt{x+1}\, dx = \int (u^2-1)\cdot u \cdot 2u\, du$$

$$= 2\int (u^2-1)\cdot u^2\, du = 2\int (u^4 - u^2)\, du$$

$$= 2\int u^4\, du - 2\int u^2\, du = 2\frac{u^5}{5} - 2\frac{u^3}{3} + c$$

$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$$

* مثال 5 بعد الورقة