**Eigenvalues and Eigenvectors**

 Given a square matrix A, suppose there are a constant $λ$ and a nonzero vector x such that

$$AX=λX$$

 then$ λ$ is called an Eigenvalue of A, and$ X$ is an Eigenvector of A corresponding to$ λ$. The eigenvalues and the corresponding eigenvectors always exist for any given square matrix.

**Defiition1**

***Let***$ A$ ***be a***$n nxn$ ***matrix. There is a number*** $λ ϵ C$ ***and a vector*** $X\ne \vec{0}$ ***such that*** $AX=λX. $ ***We say that***$ λ$ ***is an Eigenvalue of A, and***$ X$ ***is an Eigenvector of A****.*

**Example 1**

$$A=\left[\begin{matrix}1&3\\6&-2\end{matrix}\right], X=\left[\begin{matrix}x\_{1}\\x\_{2}\end{matrix}\right],$$

then

$$AX=λX or AX-λX=\vec{ 0} or \left(A-λΙ\right)X=\vec{ 0},$$

Where $Ι=\left[\begin{matrix}1&0\\0&1\end{matrix}\right], \vec{ 0}=\left[\begin{matrix}0\\0\end{matrix}\right] $

$$⟹ \left(\left[\begin{matrix}1&3\\6&-2\end{matrix}\right]-λ\left[\begin{matrix}1&0\\0&1\end{matrix}\right]\right)\left[\begin{matrix}x\_{1}\\x\_{2}\end{matrix}\right]=\left[\begin{matrix}0\\0\end{matrix}\right].$$

$$⟹ \left[\begin{matrix}1-λ&3\\6&-2-λ\end{matrix}\right]\left[\begin{matrix}x\_{1}\\x\_{2}\end{matrix}\right]=\left[\begin{matrix}0\\0\end{matrix}\right].$$

The last linear system has a non-trivial solution $X\ne \vec{ 0}$ if and only if

det$\left[\begin{matrix}1-λ&3\\6&-2-λ\end{matrix}\right]=0. $

So that $\left|\begin{matrix}1-λ&3\\6&-2-λ\end{matrix}\right|=λ^{2}+λ-20=0,$

$$⟹ λ\_{1}=-5, λ\_{2}=4 . $$

To find the eigenvector $X=\left[\begin{matrix}x\_{1}\\x\_{2}\end{matrix}\right]$, for $λ\_{1}=-5,$ we have to solve the following system of equations:

$$\left(A-λ\_{1}Ι\right)X=\vec{0},$$

$$\left[\begin{matrix}1-(-5)&3\\6&-2-(-5)\end{matrix}\right]\left[\begin{matrix}x\_{1}\\x\_{2}\end{matrix}\right]=\left[\begin{matrix}0\\0\end{matrix}\right],$$

$$\left[\begin{matrix}6&3\\6&3\end{matrix}\right]\left[\begin{matrix}x\_{1}\\x\_{2}\end{matrix}\right]=\left[\begin{matrix}0\\0\end{matrix}\right], $$

We have only one equation with two unknowns.

 $6x\_{1}+3x\_{2}=0,$

 then $x\_{1}=-\frac{3}{6}x\_{2}=-0.5x\_{2}.$

let $x\_{2}=1,$ then $x\_{1}=-0.5.$

$⟹X=\left[\begin{matrix}x\_{1}\\x\_{2}\end{matrix}\right]=\left[\begin{matrix}-0.5\\1\end{matrix}\right]$.

Similarly, we can show that the eigenvector for $λ\_{1}=4$ is

$$X=\left[\begin{matrix}x\_{1}\\x\_{2}\end{matrix}\right]=\left[\begin{matrix}-1\\1\end{matrix}\right].$$

**Defiition2**

 ***The equation*** $det\left(A-λ\_{1}Ι\right)=0$***, called the characteristic equation of a square matrix*** $A.$

**Defiition1**

***The eigenvalues of a square matrix*** $A$ ***are the roots of the characteristic***

***equation*** $det\left(A-λ\_{1}Ι\right)=0$*.*

**Example 1**

Find Eigenvalue of and$ the$ Eigenvector of the matrix $A=\left[\begin{matrix}2&3\\4&3\end{matrix}\right].$

*Solution*

the characteristic equation is det$\left[\begin{matrix}2-λ&3\\4&3-λ\end{matrix}\right]=λ^{2}-5λ-6=0.$

The eigenvalues are, therefore, r = −1 and 6.

$λ=\frac{5\mp \sqrt{5^{2}-4(-6)}}{2}$, or

$\left(λ+1\right)\left(λ-6\right)=0$

$$⟹ λ\_{1}=-1, λ\_{2}=6 . $$

Next, we will substitute each of the two eigenvalues into the matrix equation $\left(A-λ\_{1}Ι\right)X=\vec{0}$. For $λ\_{1}=-1$, the system of linear equations is

$$\left[\begin{matrix}2-(-1)&3\\4&3-(-1)\end{matrix}\right]\left[\begin{matrix}x\_{1}\\x\_{2}\end{matrix}\right]=\left[\begin{matrix}0\\0\end{matrix}\right]$$

$$\left[\begin{matrix}3&3\\4&4\end{matrix}\right]\left[\begin{matrix}x\_{1}\\x\_{2}\end{matrix}\right]=\left[\begin{matrix}0\\0\end{matrix}\right],$$

$$\begin{matrix}3x\_{1}+3x\_{2}=0\\4x\_{1}+4x\_{2}=0,\end{matrix},$$

Notice that the matrix equation represents a degenerated system of two linear equations. Both equations are constant multiples of the equation

$$x\_{1}+x\_{2}=0,$$

 There is now only one equation for the 2 unknowns, therefore, there are infinitely many possible solutions. This is always the case when solving for eigenvectors. Necessarily, there are infinitely many eigenvectors corresponding to each eigenvalue.

Solving the equation

$$x\_{1}+x\_{2}=0,$$

we get the relation $$x\_{1}=-x\_{2}.$$

 Hence, the eigenvectors corresponding to$ λ=-1$ are all nonzero multiples of

$K\_{1}=\left[\begin{matrix}1\\-1\end{matrix}\right]$.

Similarly , the eigenvectors corresponding to$ λ=6$ are all nonzero multiples of

$K\_{1}=\left[\begin{matrix}3\\4\end{matrix}\right]$.

**Note**

If A is any 2 × 2 matrix, then its characteristic equation is

$det\left[\begin{matrix}a-λ&b\\c&d-λ\end{matrix}\right]=λ^{2}-\left(a+d\right)λ+(ad-cb)=0$.

$$trA=a+d, detA=ad-cb. $$

*So that*

$λ=\frac{trA\mp \sqrt{trA^{2}-4detA}}{2}$,