CHAPTER SIX
(SIMILARITY THEORY TO STUDY PROPERTIES OF BOUNDARY LAYER)

7-1 Dimensional analysis :

Dimensional analysis is a simple but powerful method of investigating a variety of scientific phenomena and establishing useful relationships between the various quantities or parameters, based on their dimensions. One can define a set of fundamental dimensions, such as length [L], time [T], mass [M], etc., and express the dimensions of all the quantities involved in terms of these fundamental dimensions. A representation of the dimensions of a quantity or a parameter in terms of fundamental dimensions constitutes a dimensional formula, e.g., the dimensional formula for fluid viscosity is $[\mu] = [ML^{-1}T^{-1}]$. If the exponents in the dimensional formula are all zero, the parameter under consideration is dimensionless. One can form dimensionless parameters from appropriate combinations of dimensional quantities; e.g., the Reynolds number $Re = VL\rho\backslash\mu$ is a dimensionless combination of fluid velocity V, the characteristic length scale L, density ρ, and viscosity μ..

Dimensionless groups or parameters are of special significance in any dimensional analysis in which the main objective is to seek certain functional relationships between the various dimensionless parameters. There are several reasons for considering dimensionless groups instead of dimensional quantities or variables. First, mathematical expressions of fundamental physical laws are dimensionally homogeneous (i.e., all the terms in an expression or equation have the same dimensions) and can be written in dimensionless forms simply by an appropriate choice of scales for normalizing the various quantities. Second, dimensionless relations represented in mathematical or graphical form are independent of the system of units used and they facilitate comparisons between data obtained by different investigators at different locations and times. Third, and, perhaps, the most important reason for working with dimensionless parameters is that nondimensionalization always reduces the number of parameters that are involved in a functional relationship. This follows from the well-known Buckingham Pi theorem, which states that if m quantities (Q_1! Q_2, • • •, Q_m), involving n fundamental dimensions, form a dimensionally homogeneous equation, the relationship can always be expressed in terms of $m — n$ independent dimensionless groups (Π_1 , Π_2 , …., Π_{m-n}) made of the original m quantities. Thus, the dimensional functional relationship

$$f(Q_1, Q_2, \dots\dots, Q_m) \dots\dots\dots\dots\dots\dots\dots$$

Is equivalent to dimensionless relation

$$F(\Pi_1, \Pi_2, \dots, \Pi_{m-n}) = 0$$

In particular, when only one dimensionless group can be formed out of all the quantities, i.e., when $m — n= 1$, that group must be a constant, since it cannot be a function of any other parameters. If there are two II-groups, one must be a unique function of the other, and so on. Dimensional analysis does not give actual forms of the functions F, F_1, etc., or values of any dimensionless constants that might result from the analysis. This must be obtained by other means, such as further theoretical considerations and experimental observations. It is common practice to follow dimensional analysis by a systematic experimental study of the phenomenon to be investigated .

7-2 Similarity Theory

The Buckingham Pi theorem and dimensional analysis discussed above are merely mathematical formalisms and do not deal with the physics of the problem. The actual formulation of a similarity theory involves several steps, some of which require physical intuition, other theoretical considerations, prior observational information, and possibly new experiments designed to test the theory. The five steps involved in developing and testing a similarity theory are:

(1) Define the scope of the theory with all the restrictive assumptions clearly stated.

(2) Select an optimal set of relevant independent variables on which one or more variable of interest may depend. This constitutes a similarity hypothesis about the functional dependence between the various variables. Only one dependent variable is considered at a time in such a functional relation¬ship, but one can have several functional relationships (e.g., one for each dependent variable).

(3) Perform dimensional analysis after determining the number of possible independent dimensionless groups and organize the variables into dimen-sionless groups.

(4) Express functional relationships between dimensionless groups, one of which should contain the dependent variable. These constitute the similarity relations or similarity theory predictions (one for each dependent variable).

(5) Gather relevant data from previous experiments that satisfy the restrictive assumptions of the similarity theory, or perform a new experiment to test the initial similarity hypothesis and similarity theory predictions. Experimental data will tell us whether the original similarity hypothesis is correct or not. If the theory is verified by experimental data, the latter can also be used to determine empirical forms of the various similarity functions by appropriate curve fitting through data plots.

The ultimate result of this five-step procedure is a set of empirical equations or fitted curves through plots of experimental data, all involving dimensionless similarity parameters. For a successful similarity theory, verified by experiments, the empirical similarity relations are expected to be universal and can be used at other locations with different surface characteristics and under different meteorological conditions.

In the first step, certain restrictive assumptions are made in order to reduce the number of independent variables involved in a similarity hypothesis, so that one can reduce the number of dimensionless parameters to a minimum that is consistent with the physics. The fewer the dimensionless parameters, the more powerful are the similarity theory predictions, and the easier it is to verify them .

and to determine empirical similarity relations or constants from carefully conducted experiments and observations. For example, the most commonly used simplifying assumptions in all the proposed PBL similarity theories are:

(1) mean flow is steady or stationary;

(2) mean flow is horizontally homogeneous, implying a flat and homogeneous surface;

(3) viscosity and other molecular diffusivities are not relevant in the bulk of the PBL flow outside molecular sublayers.

(4) near-surface canopy variables can be ignored in the formulation of a similarity theory for the horizontally homogeneous part of the PBL.

Additional assumptions are sometimes made depending on the type of the PBL (e.g., for a barotropic PBL, geostrophic winds are independent of height) and its stability regime (e.g., under convective conditions, shear effects and the Coriolis parameter are ignored). In surface-layer similarity theories, the Coriolis parameter, geostrophic winds and shears, and the PBL height are all considered irrelevant.

The second step, involving the selection of independent variables in the formulation of a similarity hypothesis, is the most crucial step in the development of a successful similarity theory. On the one hand, one cannot ignore any of the important variables or parameters on which the dependent variable really depends, because this might lead to a completely wrong or unphysical relationship. On the other hand, if unnecessary and irrelevant variables are included in the original similarity hypothesis, they will unnecessarily complicate the analysis and make empirical determination of the various functional relationships extremely difficult, if not impossible. In principle, experimental data should tell us if there are irrelevant variables or similarity parameters, which can be dropped from the similarity theory without any significant loss of generality. It is always desirable to keep the number of independent variables to a minimum, consistent with physics. Sometimes, it may become necessary to break down the domain of the problem or phenomenon under investigation into several small subdomains, so that simpler similarity hypotheses can be formulated for each of them separately. For example, the atmospheric boundary layer is usually divided into a surface layer and an outer layer or a mixed layer for dimensional analysis and similarity considerations.

The third step is straightforward, once the number of independent dimensionless groups is determined by the Buckingham's Pi theorem. But, there is always some flexibility and choice in formulating dimensionless parameters. For this, it is often convenient to first determine the appropriate scales of length, velocity, etc., from the independent variables. Then, dimensionless parameters can be determined merely by inspection. If there are more than one length or velocity scales, their ratio forms a dimensionless similarity parameter.

The similarity relations or predictions in the fourth step are simply expressions of dimensionless groups containing dependent variables as unspecified functions of other dimensionless (similarity) parameters. If there is no similarity parameter that can be formed by independent variables only, the dependent parameter (II-group) must simply be a constant.

Finally, a thorough experimental verification of the similarity hypothesis and resulting similarity relations is necessary before a proposed similarity theory becomes widely accepted and successful. For this, experiments are conducted at different locations, under more or less idealized conditions implied in the theory.

However, the whole wide range of parameters should be covered. Sometimes, experimental data can verify some of the similarity relations, but not others, in which case the theory is considered only partially successful with applicability to only specified variables .

7-3 The M-O similarty Theory:

The following characteristic scales of length, velocity, and temperature are used to form dimensionless groups in the Monin-Obukhov similarity theory:

length scales : z and L
velocity scale : u*
temperature scale : $\theta = -\frac{H}{\rho\, C_p\, u_*}$

The similarity prediction that fellows from the M-O hypothesis is that any mean flow or average turbulence quantity in the surface layer, when normalized by an appropriate combination of the above-mentioned scales, must be a unique function of z/L only. Thus, a number of similarity relations can be written for the various quantities (dependent variables) of interest. For example, with the **x** axis oriented parallel to the surface stress or wind (the appropriate surface layer coordinate system), the dimensionless wind shear and potential temperature gradient are usually expressed as

$$\left(\frac{kz}{u_*}\right)\left(\frac{\partial u}{\partial z}\right) = \Phi_m(\zeta).$$

$$\left(\frac{kz}{\theta_*}\right)\left(\frac{\partial \theta}{\partial z}\right) = \Phi_h(\zeta) \qquad .$$

in which the von Karman constant k is introduced only for the sake of convenience, so that $\Phi_m(0) = 1$, and $\Phi_m(\zeta)$ and $\Phi_h(\zeta)$ are the basic universal similarity functions which relate the constant fluxes

$$\tau = \tau_0 = \rho u_*^2 \qquad (11.3)$$

$$\mathrm{H} = \mathrm{H}_0 = -\rho c_p \mathrm{u}_* \theta_*$$

to the mean gradients in the surface layer.
It is easy to show from the definition of Richardson number and Eqs. (! 1 . 1 . 2 ل)-(ل) that

Ri = 11.4غ،ج(ة)ةصا) ()

which relates Ri to the basic stability parameter ع = z/L of the M-0 similarity theory. The inverse of Eq. (11.4), namely, ع = /(Ri) is often used to determine ع

and, hence, the Obukhov length L from easily measured gradients of velocity and temperature at one or more heights in the surface layer. Then, Eqs. (11.2) and (11.3) can be used to determine the fluxes of momentum and heat, knowing the empirical forms of the similarity functions $4>_m(£)$ and 0h(£) from carefully conducted experiments.
Other properties of flow which relate turbulent fluxes to the local mean gradients, such as the exchange coefficients of momentum and heat and the corresponding mixing lengths, can also be expressed in terms of

7-4 Examples of similarity theories

In order to illustrate the method and usefulness of dimensional analysis and similarity, let us consider the possible relationship for the mean potential temperature gradient $(d\theta/dz)$ as a function of the height (z) above a uniform heated surface, the surface heat flux (Ho), the buoyancy parameter (g/T_0) which appears in the expressions for static stability and buoyant acceleration, and the relevant fluid properties (ρ and C_p) in the near-surface layer when free convection dominates any mechanical mixing (this latter condition permits dropping of all shear-related parameters from consideration). To establish a functional relationship in the dimensional form :

$$f\left(\frac{d\theta}{dz}, H_0, \frac{g}{T_0}, z, \rho, c_p\right) = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. (9.34)$$

would require extensive observations of temperature as a function of height and surface heat flux at different times and locations (to represent different types of surfaces and radiative regimes). If the relationship is to be further generalized to other fluids, laboratory experiments using different fluids will also be necessary. Considerable simplification can be achieved, however, if ρ and C_p are combined with H_0 into what may be called kinematic heat flux $\frac{H_0}{\rho c_p}$, so that Equation (9.34) can be written as

$$F\left(\frac{d\theta}{dz}, \frac{H_0}{\rho c_p}, \frac{g}{T_0}, z\right) = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (9.35)$$

If we now use the method of dimensional analysis, realizing that only one dimensionless group can be formed from the above quantities, we obtain

$$\left(\frac{\partial\theta}{\partial z}\right)\left(\frac{H}{\rho C_P}\right)^{-\frac{2}{3}}\left(\frac{g}{T}\right)^{\frac{1}{3}} = C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9.36)$$

in which the left-hand side is the dimensionless group that is predicted to be a constant. The value of the constant C can be determined from only one carefully conducted experiment, although a thorough experimental verification of the above relationship might require more extensive observations.

The desired dimensionless group from a given number of quantities can often be formed merely by inspection. A more formal and general approach would be to write and solve a system of algebraic equations for the exponents of various quantities involved in the dimensionless group. For example, the dimensionless group formed out of all the parameters in Equation (9.35) may be assumed as

$$\Pi = \left(\frac{\partial\theta}{\partial z}\right)\left(\frac{H_0}{pc_p}\right)^a\left(\frac{g}{T_0}\right)^b z^c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9.37)$$

in which we have arbitrarily assigned a value of unity to one of the indices (here, the exponent of $\frac{\partial\theta}{\partial z}$) , because any arbitrary power of a dimensionless quantity is also dimensionless. Writing Equation (9.37) in terms of our chosen fundamental dimensions (length, time, and temperature), we have

$$[L^{\circ}T^{\circ}K^{\circ}] = [K\,L^{-1}][KLT^{-1}]^a[LT^{-2}K^{-1}]^b[L]^c$$

from which we obtain the algebraic equations

$$\begin{aligned} 0 &= -1 + a + b + c \\ 0 &= -a - 2b \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9.38) \\ 0 &= l + a - b \end{aligned}$$

whose solution gives a = —2/3, b = 1/3, and c = 4/3. Substituting these values in Equation (9.37) and equating the only dimensionless group to a constant, then, yields Equation (9.36).

Another approach is to first formulate the characteristic scales of length, velocity, etc., from combinations of independent variables and then use these scales to normalize the dependent variables. In the case of multiple scales, their ratios form the independent dimensionless groups. In the above example of temperature distribution over a heated surface, with ∂θ/∂z as the dependent variable and the remaining quantities as independent variables, the following scales can be formulated out of the latter:

$$length: \qquad z$$

$$temperature: \qquad \theta_f = \left(\frac{H}{\rho\, C_P}\right)^{2/3} \left(\frac{g}{T_0}\right)^{-1/3} z^{-1/3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9.41)$$

$$velocity \qquad u_f = \left(\frac{H_0\, g}{\rho cp\, T_0}\, z\right)^{1/3}$$

Then, the appropriate dimensionless group involving the dependent variable is (z/θ_f) (∂θ/∂z), which must be a constant, since no other independent dimensionless groups can be formed from independent variables. This procedure also leads to Equation (9.36); it is found to be more convenient to use when a host of dependent variables are functions of the same set of independent variables. For example, standard deviations of temperature and vertical velocity fluctuations in the free convective surface layer are given by

Equations (9.41) have proved to be quite useful for the atmospheric surface layer under daytime unstable conditions and are supported by many observations (Monin and Yaglom, 1971, Chapter 5; Wyngaard, 1973) from which ce ^ 1.3 andCw ^ 1.4. The above similarity theory was originally proposed by Obukhov and is more commonly known as the local free convection similarity theory. Local free convection similarity and scaling are not found to be applicable to horizontal velocity fluctuations, so that au and av are not simply proportional to wf. Instead, they are found to be strongly influenced by large-eddy motions (convective updrafts and downdrafts) which extend through the whole depth of the CBL. Since the PBL height is ignored in the local free convection similarity theory, its applicability is limited to vertical velocity and temperature fluctuations.

If the boundary layer height (h) is also added to the list in Equation (9.35), we would have, according to the pi theorem, two independent dimensionless

groups and the predicted functional relationship would be

to be functions of zjh. However, subsequent analysis of experimental data might indicate that the dependence of these on zjh is very weak or nonexistent and, hence, the irrelevance of h in the original hypothesis. The inclusion of h would be quite justified and even necessary, however, if we were to investigate the turbulence structure of the mixed layer, or au and av even in the surface layer. For example, the mixed-layer similarity hypothesis proposed by Deardorff (1970b) states that turbulence structure in the convective mixed layer depends on z, g/TQ, HQ/pcp, and h. Then, the relevant mixed-layer similarity scales are:

The corresponding similarity predictions are that the dimensionless structure parameters cu/W*, GW/W*, etc., must be some unique function of zjh. This mixed-layer similarity theory has proved to be very useful in describing turbulence and diffusion in the convective boundary layer (Arya, 1999).
As the number of independent variables and parameters is increased in a similarity hypothesis, not only the number of independent dimensionless groups increases, but also the possible combinations of variables in forming such groups become large. The possibility of experimentally determining their functional relationships becomes increasingly remote as the number of dimen¬sionless groups increases beyond two or three. A generalized PBL similarity theory will have to include all the possible factors influencing the PBL under the whole range of conditions encountered and, hence, might be too unwieldy for practical use. Simpler PBL similarity theories and scaling will be described in Chapter 13, while the surface layer similarity theory/scaling will be discussed in more detail in Chapters 10-12.

Example Problem 2

In a convective boundary layer during the midday period when TQ = 300 K, HQ = 500 W m~2, and h = 1500 m, calculate and compare the standard deviations of vertical velocity and temperature fluctuations at the 10 m and 100 m height levels.

Solution

Using the local free-convection similarity relations (9.41) with Cg = 1.3, Cw = 1.4, and pcp = 1200 J K"1 m~3, the computed values oiaw and ag at the two heights in the convective surface layer (z < 0.1/z = 150 m) are as follows:

Height, z (m)	aw (m s	
10	0.72	1.05
100	1.55	0.49

Note that aw increases and og decreases by a factor of lO1^3 = 2.15 as the height increases from 10 to 100 m.

Problems

Q1) suppose that wind speed , $\overline{M}$, near the surface at night is a function of $\frac{g}{\overline{\theta}}$, $\overline{w'\theta'}$, $\frac{\partial Z}{\partial x}$, $\frac{\overline{\partial \theta}}{\partial Z}$, z_0 and $\overline{U}_g$. use Buckingham PI dimensional analysis to determine the relevant PI groups .

Q2) assume that the following variables are relevant for flow over an isolated hill:

$\left(\frac{g}{\theta}\right) \frac{\overline{\partial \theta_v}}{\partial Z} = stability\ parameter$

$\overline{M}$ = wind speed

D = diameter of hill

H = height of hill

Use Buckinham PI Methods to find the dimensionless groups for this problem .

Q3) during the night , assume that the TKE is a function of the following parameters : (z_0 , z , $\frac{g}{\overline{\theta}}$, $\Delta\theta$, $\overline{U}_g$, ν) where ν has units of length times velocity . use dimensional analysis to find the dimensionless PI , groups .

Q4) given the following variables and their dimensions :

z	$height$	L
$\frac{g}{\theta}$	$buoyancy$	$L\ T-2\ K-1$
TKE	$turbulent\ \ tk\ per\ unit\ mass$	$L^2\ T^{-2}$
z_i	$depth\ of\ the\ mixed\ layer$	L
$w'\theta'$	$surface\ kinematic\ heat\ flux$	$L\,T^{-1}\,K$

Perform a dimensional analysis to find PI groups for z and tke , using the remaining variables as the primary variables .