Al-Mustansiriyah University College of Science Dept. of Math.

Finite Mathematics I-(2015-2016)

Course Outline
First Semester

Course Title: Finite Mathematics I

Instructor: Assistant Professor Dr. Emad Bakr Al-Zangana

Stage: The First

Contents

Chapter 1	Linear equations	Summation notation, linear equations, system of linear equations, elimination methods, substitution methods.
Chapter 2	Matrix Algebra	Matrix definition, some types of matrices
Chapter 3	Operation on Matrices	Addition and multiplication of matrices and theorems on it, (reduce) row echelon form.
Chapter 4	The Inverse of a Matrix	Definition and ways of computing matrix inverse
Chapter 5	Determinants	Definition, properties and ways of computing
Chapter 6	Solution of Linear systems	Gauss-Jordan reduction, Gauss elimination, Cramer's Rule, matrix inverse

Chapter One
Linear Equations

Definition 1.1: (Summation Notation)
By $\sum_{i=1}^{n} a_i$ we mean $a_1 + a_2 + \cdots a_n$. That is,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots a_n \ .$$

The letter i is called the **index of summation.**
By $\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}$ we mean that we first sum on j and then sum the resulting expression on i.

Example 1.2: Let $a_1 = 10$, $a_2 = 12$. Then,

$$\sum_{i=1}^{2} a_i = a_1 + a_2 = 10 + 12 = 32.$$

Theorem 1.3:
1- $\sum_{i=1}^{n}(r_i + s_i)a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i$.
2-$\sum_{i=1}^{n} c(r_i a_i) = \sum_{i=1}^{n}(cr_i)a_i = c(\sum_{i=1}^{n} r_i a_i)$.
3-$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m}\sum_{i}^{n} a_{ij}$.

Proof :
(1) and (2) exercise.
(3) $\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sum_{i=1}^{n}(a_{i1} + a_{i2} + \cdots + a_{im})$

$$= (a_{11} + a_{12} + \cdots + a_{1m}) + (a_{21} + a_{22} + \cdots + a_{2m}) + +(a_{n1} + a_{n2} + \cdots + a_{nm})$$

$$= (a_{11} + a_{21} + \cdots + a_{n1}) + (a_{12} + a_{22} + \cdots + a_{n2}) + +(a_{1m} + a_{2m} + \cdots + a_{nm})$$

$$= \sum_{j=1}^{m}(a_{1j} + a_{2j} + \cdots + a_{nj}) = \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij} .$$

Example 1.4:

$$\sum_{i=1}^{3}\sum_{j=1}^{2} a_{ij} = \sum_{i=1}^{3}(a_{i1} + a_{i2})$$

$$= (a_{11} + a_{12}) + (a_{21} + a_{22}) + (a_{31} + a_{32})$$

$$= (a_{11} + a_{21} + a_{31}) + (a_{12} + a_{22} + a_{32})$$

$$\sum_{j=1}^{2}(a_{j1} + a_{j1} + a_{j3}) = \sum_{j=1}^{2}\sum_{i=1}^{3} a_{ij} .$$

Systems of Linear Equations

Definition 1.5: An equation containing ***n*** variables is said to be **linear** if it can be written in the form

$$b = a_1x_1 + a_2x_2 + \cdots + a_nx_n \text{ --------> } \mathbf{(1)}$$
$$= \sum_{i=1}^{n} a_ix_i.$$

Where $x_1, x_2, \cdots, x_n$ are *n* distinct variables, $a_1, a_2, \cdots, a_n$, *b* are constants, and at least one of the $a_i{}'$s is not 0.

Definition 1.6: A solution to a linear equation **(1)** is a sequence of ***n*** numbers $s_1, s_2, \cdots, s_n$, which has the property that **(1)** is satisfied when

$$x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$$

are substituted in **(1).**

Example 1.7:

(i) the equation $6x_1 - 3x_2 + 4x_3 = -13$ is linear equation of three variables.

$x_1 = 2,\ x_2 = 3,\ \ x_3 = -4$ is a solution to the linear equation

$$6 \cdot 2 - 3 \cdot 3 + 4 \cdot (-4) = -13.$$

This is not the only solution to the given linear equation, since

$$x_1 = 3, \qquad x_2 = 1, \quad x_3 = -7$$

is another solution.

(ii)

Consider the equation

$$2x + y = 4$$

If we substitute $x = -2$ in the equation, we obtain

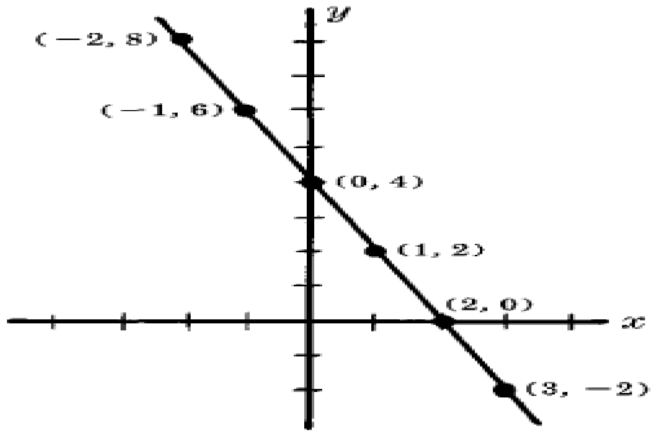
$$2 \cdot (-2) + y = 4 \qquad \text{or} \qquad -4 + y = 4 \qquad \text{or} \qquad y = 8$$

Hence $(-2, 8)$ is a solution. If we substitute $x = 3$ in the equation, we obtain

$$2 \cdot 3 + y = 4 \qquad \text{or} \qquad 6 + y = 4 \qquad \text{or} \qquad y = -2$$

Hence $(3, -2)$ is a solution. The table on the right lists six possible values for x and the *corresponding* values for y, i.e. six solutions of the equation.

x	y
-2	8
-1	6
0	4
1	2
2	0
3	-2



Graph of $2x + y = 4$

Definition 1.8: A system of ***m*** linear equations in ***n*** unknowns, or a linear system, is a set of ***m*** linear equations each in ***n*** unknowns and it is of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

The numbers a_{ij} are called the **coefficients** of x_j and b_i is called the **constant term** for each i.

A solution to a linear system is a sequence of ***n*** numbers $s_1, s_2, \cdots, s_n$, which has the property that each equation in the system is satisfied when

$$x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$$

are substituted in the system.

Definition 1.9:

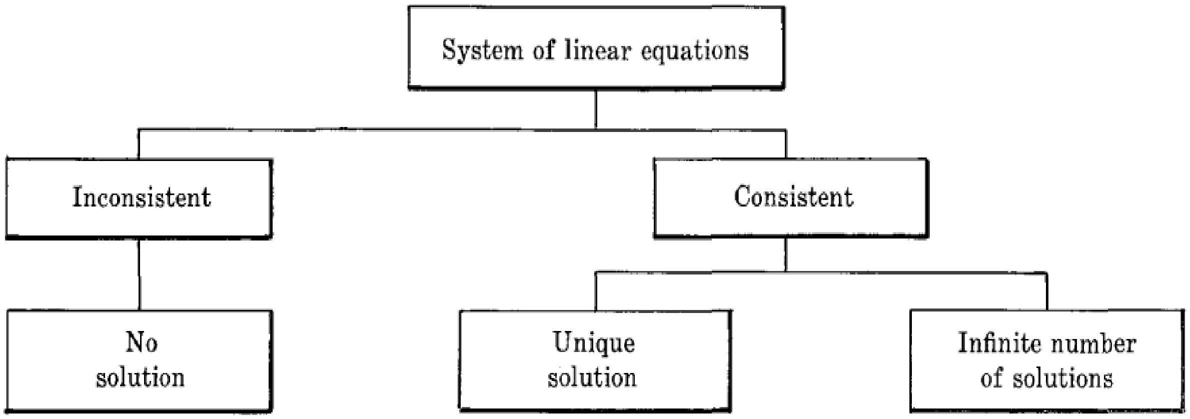
(i) A system which each constant term is zero called **homogenous system**.

(ii) If a system of equations has at least one solution, it is said to be **consistent**.

(iii) If it has no solution, it is said to be **inconsistent**.

(iv) If a consistent system of equations has exactly one solution, the equations of the system are said to be **independent**.

(v) If it has an infinite number of solutions, the equations are called **dependent**.



Solving by Substitution

1.10: Steps for Solving by Substitution

STEP 1: Pick one of the equations and solve for one of the variables in terms of the remaining variables.
STEP 2: Substitute the result in the remaining equations.
STEP 3: If one equation in one variable results, solve this equation. Otherwise, repeat Steps 1 and 2 until a single equation with one variable remains.
STEP 4: Find the values of the remaining variables by back-substitution.

Example 1.11: (1) Solve

$$\begin{cases} 2x + y = 5 & (1) \\ -4x + 6y = 12 & (2) \end{cases}$$

Solution: Solve the first equation for ***y***, obtaining

$$2x + y = 5 \qquad (1)$$

$$y = -2x + 5 \qquad \text{Subtract } 2x \text{ from each side}$$

Substitute this value of ***y*** in the second equation. This results in an equation containing one variable, which we can solve.

$$-4x + 6y = 12 \quad (2)$$
$$-4x + 6(-2x + 5) = 12 \quad \text{Substitute } y = -2x + 5 \text{ in (2)}$$
$$-4x - 12x + 30 = 12 \quad \text{Remove parentheses.}$$
$$-16x = -18 \quad \text{Combine like terms; subtract 30 from each side.}$$
$$x = \frac{-18}{-16} = \frac{9}{8} \quad \text{Divide each side by } -16.$$

by substituting for $x = \frac{9}{8}$ in one of the original equations we get that $y = \frac{11}{4}$.
We can also write the solution as the ordered pair

$$S.S. = \left\{\left(\frac{9}{8}, \frac{11}{4}\right)\right\}.$$

The system is consistent and independent.

Solve Systems of Equations by Elimination

1.12: Rules for Obtaining an Equivalent System of Equations

1- Interchange any two equations in the system.

2- Multiply (or divide) each side of an equation by the same nonzero constant.

3- Replace any equation in the system by the sum (or difference) of that equation and a nonzero multiple of any other equation in the system.

Example 1.13: (1) Solve

$$\begin{array}{rcl} 2x + 3y & = 1 & ---\rightarrow (1) \\ -x + y & = -3 & ---\rightarrow (2) \end{array}.$$

Solution: Multiply each side of Equation (2) by 2 so that the coefficients of x in the two equations are opposites of one another. The result is the equivalent system

$$\begin{cases} 2x + 3y = 1 & (1) \\ -2x + 2y = -6 & (2) \end{cases}$$

Now replace Equation (2) of this system by the sum of the two equations, to obtain an equation containing just the variable y, which we can solve.

$$\begin{cases} 2x + 3y = 1 & (1) \\ -2x + 2y = -6 & (2) \end{cases}$$
$$5y = -5 \quad \text{Add (1) and (2).}$$
$$y = -1 \quad \text{Solve for } y.$$

Back-substitute this value for $\boldsymbol{y}$ in Equation (1) and simplify to get

$$2x + 3y = 1 \quad (1)$$
$$2x + 3(-1) = 1 \quad \text{Substitute } y = -1 \text{ in (1).}$$
$$2x = 4 \quad \text{Simplify.}$$
$$x = 2 \quad \text{Solve for } x.$$

The solution of the original system is $x = 2, y = -1$, or using ordered pairs
$$S.S = \{(2, -1)\}.$$
The system is consistent and independent.

(2) Use the method of elimination to solve the system of equations.

$$\begin{cases} x + y - z = -1 & (1) \\ 4x - 3y + 2z = 16 & (2) \\ 2x - 2y - 3z = 5 & (3) \end{cases}.$$

Solution: For a system of three equations, we attempt to eliminate one variable at a time, using pairs of equations, until an equation with a single variable remains.

Use Equation (1) to eliminate the variable x from Equations (2) and (3).

$$\begin{array}{llrr} x + y - z = -1 & \text{(1) Multiply by } -4. & -4x - 4y + 4z = 4 & (1) \\ 4x - 3y + 2z = 16 & (2) & \underline{4x - 3y + 2z = 16} & (2) \\ & & -7y + 6z = 20 & \text{Add.} \end{array}$$

So, we have new equation $-7y + 6z = 20 - --\rightarrow (4)$.

$$\begin{array}{llrr} x + y - z = -1 & \text{(1) Multiply by } -2. & -2x - 2y + 2z = 2 & (1) \\ 2x - 2y - 3z = 5 & (3) & \underline{2x - 2y - 3z = 5} & (3) \\ & & -4y - z = 7 & \text{Add.} \end{array}$$

So, we have new equation $-4y - z = 7 - --\rightarrow (5)$.
Now, to eliminate z using equations (4) and (5).

$$\begin{aligned} -7y + 6z &= 20 \quad (4) \\ -4y - z &= 7 \quad (5) \text{ Multiply by 6.} \end{aligned} \qquad \begin{aligned} -7y + 6z &= 20 \quad (4) \\ -24y - 6z &= 42 \quad (5) \\ \hline -31y &= 62 \quad \text{Add.} \end{aligned} \qquad \begin{cases} x + y - z = -1 & (1) \\ -7y + 6z = 20 & (4) \\ -31y = 62 & (6) \end{cases}$$

Now solve Equation (6) for y by dividing both sides of the equation by -3.
So, $y = -2$. Back-substitute in Equation (4) and solve for z, we get $z = 1$.
Finally, back-substitute $y = -2$ and $z = 1$ in Equation (1) and solve for x we get $x = 2$.
The solution of the original system is $x = 2$, $y = -2$, $z = 1$ or, using ordered triplets, $S.S = \{(2, -2, 1)\}$. The system is consistent and independent.

(3) Solve

$$\begin{aligned} x + y + z &= 3 \quad E_1 \\ x - y - 5z &= 1 \quad E_2 \\ 2x + 3y + 5z &= 6 \quad E_3 \end{aligned}$$

Solution:
Use E_1 to eliminate z from E_2 and replace E_2 with the result.

$$\begin{aligned} 5x + 5y + 5z &= 15 \quad 5E_1 \\ x - y - 5z &= 1 \quad E_2 \\ \hline 6x + 4y &= 16 \quad E_4 \end{aligned} \qquad \text{Equivalent System} \quad \begin{aligned} x + y + z &= 3 \quad E_1 \\ 6x + 4y &= 16 \quad E_4 \\ 2x + 3y + 5z &= 6 \quad E_3 \end{aligned}$$

Use E_1 to eliminate z from E_3 and replace E_3 with the result.

$$\begin{aligned} -5x - 5y - 5z &= -15 \quad -5E_1 \\ 2x + 3y + 5z &= 6 \quad E_3 \\ \hline -3x - 2y &= -9 \quad E_5 \end{aligned} \qquad \text{Equivalent System} \quad \begin{aligned} x + y + z &= 3 \quad E_1 \\ 6x + 4y &= 16 \quad E_4 \\ -3x - 2y &= -9 \quad E_5 \end{aligned}$$

Now treat E_4 and E_5 as a system of two equations, and eliminate y.

$$\begin{array}{rl} 6x + 4y = 16 & E_4 \\ -6x - 4y = -18 & 2E_5 \\ \hline 0 = -2 & E_6 \end{array}$$

We have obtained a contradiction C!. The original system is inconsistent and has no solution

(4) Solve

$$\begin{array}{lll} 2x + 6y = -3 & \text{-----}\rightarrow & (1) \\ x + 3y = 2 & \text{------}\rightarrow & (2) \end{array}$$

Solution:

Solution by Substitution

Solve the second equation for x and substitute in the first equation.

$$\begin{aligned} x &= 2 - 3y \\ 2(2 - 3y) + 6y &= -3 \\ 4 - 6y + 6y &= -3 \\ 4 &= -3 \end{aligned}$$

Solution by Elimination

Multiply the second equation by -2 and add to the first equation.

$$\begin{array}{r} 2x + 6y = -3 \\ -2x - 6y = -4 \\ \hline 0 = -7 \end{array}$$

Both methods of solution lead to a contradiction (a statement that is false). An assumption that the original system has solutions must be false. The system has no solution. The graphs of the equations are parallel and the system is inconsistent.

(5) Solve

$$\begin{array}{lll} x - \frac{1}{2}y = 4 & \text{-----}\rightarrow & (1) \\ -2x + y = -8 & \text{-----}\rightarrow & (2) \end{array}$$

Solution:

Solution by Substitution

Solve the first equation for x and substitute in the second equation.

Solution by Elimination

Multiply the first equation by 2 and add to the second equation.

$$
\begin{aligned}
x &= \tfrac{1}{2}y + 4 \\
-2(\tfrac{1}{2}y + 4) + y &= -8 \\
-y - 8 + y &= -8 \\
-8 &= -8
\end{aligned}
\qquad\qquad
\begin{array}{r}
2x - y = 8 \\
\underline{-2x + y = -8} \\
0 = 0
\end{array}
$$

This time both solution methods lead to a statement that is always true. This means that the two original equations are equivalent. The system is dependent and has an infinite number of solutions. There are many different ways to represent this infinite solution set. For example,

$$S_1 = \{(x, y) \mid y = 2x - 8,\ x \text{ any real number}\}$$

and

$$S_2 = \{(x, y) \mid x = \tfrac{1}{2}y + 4,\ y \text{ any real number}\}$$

both represent the solutions to this system.

(6) Solve

$$
\begin{array}{l}
x + 2y - 3z = -4 \quad ------\rightarrow (E_1) \\
2x + y - 3z = 4 \quad -------\rightarrow (E_2)
\end{array}
$$

Solution:

To eliminating x, $2E_1 - E_2 = -3y + 3z = 12$.

$\Rightarrow y = z - 4$ -------->(3) where z is any real numbers.

Back-substitute in E_1 and solve for x, we get

$x + 2(z - 4) - 3z = -4 \quad \Rightarrow x = z + 4$ -------->(4)

Thus a solution to the linear system is

$$
\begin{aligned}
x &= z + 4 \\
y &= z - 4 \\
z &= \text{any real numbers.}
\end{aligned}
$$

This means that the linear system has infinitely many solutions.

Exercise

1.14

Q: Show that if the following linear systems are consistent (dependent or independent) or inconsistent.

1- $\begin{aligned} x - 3y &= -7 \\ 2x - 6y &= 7 \end{aligned}$	2- $\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$
3- $\begin{aligned} x + 2y &= 10 \\ 2x - 2y &= -4 \\ 3x + 5y &= 20 \end{aligned}$	4- $\begin{aligned} x - y + z &= 1 \\ 2x + y + z &= 6 \\ 7x - y + 5z &= 15 \end{aligned}$
5- $\begin{aligned} x - 3y &= 2 \\ 2y + z &= -1 \\ x - y + z &= 1 \end{aligned}$	6- $\begin{aligned} 2a + 4b + 3c &= -6 \\ a - 3b + 2c &= -15 \\ -a + 2b - c &= 9 \end{aligned}$
7- $\begin{aligned} 2y - z &= 2 \\ -4y + 2z &= 1 \\ x - 2y + 3z &= 0 \end{aligned}$	8- $\begin{aligned} 3x + 3y + 2z &= 4 \\ x - y - z &= 0 \\ 2y - 3z &= -8 \end{aligned}$
9- $\begin{aligned} 3x + 3y + 2z &= 4 \\ x - 3y + z &= 10 \\ 5x - 2y - 3z &= 8 \end{aligned}$	10- $\begin{aligned} 5x - y &= 13 \\ 2x + 3y &= 12 \end{aligned}$
11- $\begin{aligned} 2x + 3y &= 6 \\ x - y &= \frac{1}{2} \end{aligned}$	12- $\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &= 3 \\ \frac{1}{4}x - \frac{2}{3}y &= -1 \end{aligned}$

Q: Prove that $\sum_{i=1}^{n}(r_i + s_i + w_i + v_i)a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i + \sum_{i=1}^{n} w_i a_i + \sum_{i=1}^{n} v_i a_i$, where r_i, s_i, w_i, v_i, a_i are in R or $\mathbb{C}$.

Chapter Two
Matrix Algebra

Definition 2.1: A **matrix** is defined as a rectangular array of the form:

A **matrix** is defined as a rectangular array of the form:

$$\begin{array}{cc} & \begin{array}{ccccccc} \text{Column } 1 & \text{Column } 2 & & \text{Column } j & & \text{Column } n \end{array} \\ \begin{array}{c} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } i \\ \vdots \\ \text{Row } m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad (1)$$

The symbols $a_{11}, a_{12}, \ldots$ of a matrix are referred to as the **entries** (or **elements**) of the matrix. Each entry a_{ij} of the matrix has two indices: the **row index,** *i*, and the **column index,** *j*.

The symbols $a_{i1}, a_{i2}, \ldots, a_{in}$ represent the entries in the *i*th row, and the symbols $a_{1j}, a_{2j}, \ldots, a_{mj}$ represent the entries in the *j*th column.
If we denote the matrix in display (1) above by *A*, then we can abbreviate *A* by

$$A = [a_{ij}] \qquad i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n.$$

The matrix *A* has *m* rows and *n* columns.
Definition: The **dimension of a matrix** *A* is determined by the number of rows and the number of columns in the matrix. If a matrix *A* has *m* rows and *n* columns, we denote the dimension of *A* by $m \times n$ read as "*m* by *n*.'' $A = [a_{ij}]_{m\times n}$.

Some Types of Matrices

Definition 2.2: A matrix $A = [a_{ij}]_{m\times n}$ is called a **square matrix** if a matrix *A* has the same number of rows as it has columns; that is, $m = n$. Dimension of *A* is n

Example 2.3: $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{8} & \mathbf{6} & \mathbf{0} \\ \mathbf{-2} & \mathbf{5} & \mathbf{7} \end{bmatrix}_{3\times 3}, \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{8} & \mathbf{0} \\ \mathbf{-2} & \mathbf{7} \end{bmatrix}_{3\times 2}.$

Definition2.4: In a square matrix $A = [a_{ij}]_{n\times n}$, the entries for which $i = j$ namely
$a_{11}, a_{22}, \dots , a_{nn}$ ($i = 1,2, \cdots , n$) are the **diagonal entries** of *A* which form the **main diagonal** of *A*.

Definition 2.5:
1- A matrix whose all entries are all zero is called **zero matrix** and denoted by **0.**

$$\mathbf{0} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}_{\boldsymbol{m\times n}}$$

2- A square matrix of dimension n whose all diagonal elements are all one and every term off the main diagonal is zero is called **identity matrix** and denoted by $\boldsymbol{I_n}$.

$$\boldsymbol{I_n} = \begin{bmatrix} \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \mathbf{1} & \vdots \\ \mathbf{0} & \cdots & \mathbf{1} \end{bmatrix}_{\boldsymbol{n\times n}}$$

Definition 2.6: (Diagonal Matrix)
A square matrix $A = [a_{ij}]_{n\times n}$ for which every term off the main diagonal is zero, that is, $a_{ij} = 0$ for $i \neq j$, is called a **diagonal matrix.**

Definition 2.7: (Scalar Matrix)
A diagonal matrix $A = [a_{ij}]_{n\times n}$ for which all terms on the main diagonal are equal, that is, $a_{ij} = c$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$ is called a **scalar matrix.**

Definition 2.8: (Upper Triangular)
A square matrix $A = [a_{ij}]_{n\times n}$ is called **upper triangular** if

$$a_{ij} = 0 \text{ for } i > j.$$

Definition 2.9: (Lower Triangular)
A square matrix $A = [a_{ij}]_{n\times n}$ is called **lower triangular** if

$$a_{ij} = 0 \text{ for } i < j.$$

Definition 2.10: (Equality of Matrices)
Two matrices *A* and *B* are **equal** if they are of the same dimension and if corresponding entries are equal.

Examples 2.11:

(1) If $A = \begin{bmatrix} x+y & 2z+w \\ x-y & z-w \end{bmatrix}_{2\times 2} = \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}_{2\times 2}$ find x, y, z, w.

Solution: The statement above is equivalent to the following:

$$x+y=3, \quad x-y=1,$$
$$2z+w=5, \quad z-w=4.$$

The solution is $x=2, \quad y=1, z=3, \quad w=-1$.

(2) Let A and B be two matrices given by

$$A = \begin{bmatrix} x+y & 6 \\ 2x-3 & 2-y \end{bmatrix}_{2\times 2} \quad B = \begin{bmatrix} 5 & 5x+2 \\ y & x-y \end{bmatrix}_{2\times 2}$$

Determine if there are values of x and y so that A and B are equal.

Solution: Both A and B are 2×2 matrices so $A = B$ if

$$x+y=5 \quad (1) \qquad 6=5x+2 \quad (2)$$
$$2x-3=y \quad (3) \qquad 2-y=x-y \quad (4)$$

Here we have four equations containing the two variables x and y. From Equation (4) we see that $x=2$. From equation (1), we obtain $y=3$. But $x=2, y=3$ do not satisfy either Equation (2) or Equation (3). There are *no* values for x and y satisfying all four equations. This means A and B can never be equal.

Definition 2.12: If $A = [a_{ij}]_{m\times n}$ is a matrix, then the $n \times m$ matrix $A^T = [a_{ij}{}^T]_{n\times m}$, where

$$a_{ij}{}^T = a_{ji} \quad (1 \le i \le m, 1 \le j \le n)$$

is called the **transpose** of A. Thus the transpose of A is obtained by interchanging the rows and columns of A.

The first row of A^T is the first column of A; the second row of A^T is the second column of A; and so on.

Example 2.13: If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \qquad C = [1 \quad 0 \quad -1]$$

then

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 2 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Definition 2.14: A matrix A is called symmetric if

$$A = A^T,$$

that is, $(i,j) -$ element of $A = (j,i) -$ element of A^T.

Remark 2.15: (1) A is symmetric if it is a square for which

$$a_{ij} = a_{ji}\,.$$

(2) If A is symmetric, then the elements of A are symmetric with respect to the main diagonal of A**.**

Example 2.16: If

$$\text{(a)} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 3 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 4 & 7 \\ 3 & 7 & 5 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 5 & 0 \\ 3 & 5 & 1 & 0 \end{bmatrix}$$

then (a) is not symmetric (b) is symmetric (c) is not symmetric.

Definition 2.17: A square matrix $\boldsymbol{A}$ is called **skew symmetric** if

$$a_{ij} = -a_{ji} \text{ for all } i,j.$$

Example 2.18:

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}_{\mathbf{3\times 3}}$$

Remark 2.19: The main diagonal elements of a skew symmetric matrix are all zero.

Definition 2.20: If $A = [a_{ij}]$ is an $m \times n$ matrix, the $m \times n$ matrix obtained by replacing each element of A by its complex conjugate is called the **matrix conjugate** of A and is denoted by $\overline{A}$. That is, $\overline{A} = [\overline{a_{ij}}]$

Definition 2.21: If $A = [a_{ij}]$ is an $m \times n$ matrix, the $m \times n$ matrix obtained by the transpose of $\overline{A}$ is called the **transpose conjugate** of A and is denoted by

$$\overline{A}^T = A^{\varphi}.$$

Example 2.22:

$$\boldsymbol{A} = \begin{bmatrix} 1+i & 2 & 3i \\ -4+i & 1 & 2-6i \end{bmatrix}_{\mathbf{2\times 3}}, \quad \overline{\boldsymbol{A}} = \begin{bmatrix} 1-i & 2 & -3i \\ -4-i & 1 & 2+6i \end{bmatrix}_{\mathbf{2\times 3}},$$

$$\boldsymbol{A^{\varphi}} = \begin{bmatrix} 1-i & -4+i \\ 2 & 1 \\ -3i & 2+6i \end{bmatrix}_{\mathbf{2\times 3}}.$$

Definition 2.23: A square matrix $A = [a_{ij}]$ is called to be **Hermitian** if

$$\overline{a_{ij}} = a_{ji} \text{ for all } i, j$$

That is, $\overline{\boldsymbol{A}} = \boldsymbol{A^T}$.

Example 2.23:

$$A = \begin{bmatrix} 1 & -3i \\ 3i & 2 \end{bmatrix}_{2\times 2}.$$

Remark 2.24: The main diagonal elements of a Hermitian matrix are all real.

Definition 2.25: A square matrix $A = [a_{ij}]$ is called to be **skew Hermitian** if

$$\overline{a_{ij}} = -a_{ji} \text{ for all } i, j$$

That is, $\overline{\boldsymbol{A}} = -\boldsymbol{A^T}$.

Example 2.26:

$$\boldsymbol{A} = \begin{bmatrix} i & 4+i \\ -4+i & 6i \end{bmatrix}_{\mathbf{2\times 2}}.$$

Remark 2.27: The main diagonal elements of a skew Hermitian matrix are all not real.

Definition 2.28: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then the **trace** of A denoted by $\mathrm{Tr}(A)$, is defined as the sum of all diagonal elements of A. That is,

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}.$$

Example 2.29:

Let $A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & -5 & -5 \\ 2 & 5 & 9 \end{bmatrix}_{\mathbf{3\times 3}}$. Then $\mathrm{Tr}(A) = 2 + (-5) + 9 = 6$.

Chapter Three
Operation on Matrices

Definition 3.1: (Addition of Matrices)
We define the **sum** $\boldsymbol{A}+\boldsymbol{B}$ of two matrices A and B with the same dimension as the matrix consisting of the sum of corresponding entries from A and B. That is, if $A=\left[a_{ij}\right]_{m\times n}$, $B=\left[b_{ij}\right]_{m\times n}$ are two matrices, the sum is the $m\times n$ matrix $\boldsymbol{A}+\boldsymbol{B}=\left[a_{ij}+b_{ij}\right]$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

Definition 3.2: If A is any matrix, the **additive inverse** of A, denoted by $-A$ is the matrix obtained by replacing each number in A by its additive inverse. That is, if $A=\left[a_{ij}\right]_{m\times n}$, then $-A=\left[-a_{ij}\right]_{m\times n}$.

Definition 3.3: (Subtraction of Matrices)
We define the **difference** $\boldsymbol{A}-\boldsymbol{B}$ of two matrices A and B with the same dimension as the matrix consisting of the difference of corresponding entries from A and B. That is, if $A=\left[a_{ij}\right]_{m\times n}$, $B=\left[b_{ij}\right]_{m\times n}$ are two matrices, the **difference is** $m\times n$ matrix $\boldsymbol{A}-\boldsymbol{B}=\left[a_{ij}-b_{ij}\right]$.

Definition 3.4: (Scalar Multiplication)
Let A be an $m\times n$ matrix and let c be a real number, called a **scalar.** The product of the matrix A by the scalar c, called **scalar multiplication,** is the $m\times n$ matrix cA, whose entries are the product of c and the corresponding entries of A. That is, if $A=\left[a_{ij}\right]_{m\times n}$ then $cA=\left[ca_{ij}\right]_{m\times n}$.

Example 3.5: Suppose

$$A = \begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 0 \\ 8 & 1 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix}$$

Find: (a) $4A$ (b) $\frac{1}{3}C$ (c) $3A - 2B$

Solution:

(a) $$4A = 4\begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4\cdot 3 & 4\cdot 1 & 4\cdot 5 \\ 4(-2) & 4\cdot 0 & 4\cdot 6 \end{bmatrix} = \begin{bmatrix} 12 & 4 & 20 \\ -8 & 0 & 24 \end{bmatrix}$$

(b) $$\frac{1}{3}C = \frac{1}{3}\begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\cdot 9 & \frac{1}{3}\cdot 0 \\ \frac{1}{3}\cdot(-3) & \frac{1}{3}\cdot 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$

(c) $$3A - 2B = 3\begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix} - 2\begin{bmatrix} 4 & 1 & 0 \\ 8 & 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 3 & 3\cdot 1 & 3\cdot 5 \\ 3(-2) & 3\cdot 0 & 3\cdot 6 \end{bmatrix} - \begin{bmatrix} 2\cdot 4 & 2\cdot 1 & 2\cdot 0 \\ 2\cdot 8 & 2\cdot 1 & 2(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 & 15 \\ -6 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 8 & 2 & 0 \\ 16 & 2 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 15 \\ -22 & -2 & 24 \end{bmatrix}$$

Definition 3.6: (Multiplication of Matrices)

Let *A* denote an $m \times p$ matrix, and let *B* denote an $p \times n$ matrix. The product *AB* is defined as the $m \times n$ matrix whose entry in row *i*, column *j* is the product of the *i*th row of *A* and the *j*th column of *B.*

$$\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \cdot & \cdots & \cdot \\ a_{i1} & \cdots & a_{ip} \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & c_{ij} & \cdot \\ \cdot & \cdots & \cdot \\ c_{m1} & \cdots & c_{mn} \end{pmatrix}.$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{1p}.$

Example 3.7:

Given

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

Find each product that is defined:

(i) AB (ii) BA (iii) CD (iv) DC

Solution:

(i) $$AB = \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}} \underset{2\times 4}{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}}$$

$$= \boxed{\begin{bmatrix} (2)(1)+(1)(2) & (2)(-1)+(1)(1) & (2)(0)+(1)(2) & (2)(1)+(1)(0) \\ (1)(1)+(0)(2) & (1)(-1)+(0)(1) & (1)(0)+(0)(2) & (1)(1)+(0)(0) \\ (-1)(1)+(2)(2) & (-1)(-1)+(2)(1) & (-1)(0)+(2)(2) & (-1)(1)+(2)(0) \end{bmatrix}}$$

$$= \underset{3\times 4}{\begin{bmatrix} 4 & -1 & 2 & 2 \\ 1 & -1 & 0 & 1 \\ 3 & 3 & 4 & -1 \end{bmatrix}}$$

(ii)

$$BA = \underset{2\times 4}{\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}}$$

Product is not defined.

(iii)

$$CD = \underset{2\times 2}{\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}} \boxed{= \begin{bmatrix} (2)(1)+(6)(3) & (2)(2)+(6)(6) \\ (-1)(1)+(-3)(3) & (-1)(2)+(-3)(6) \end{bmatrix}}$$

$$= \underset{2\times 2}{\begin{bmatrix} 20 & 40 \\ -10 & -20 \end{bmatrix}}$$

(iv)

$$DC = \overset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}} \overset{2\times 2}{\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}} \boxed{= \begin{bmatrix} (1)(2)+(2)(-1) & (1)(6)+(2)(-3) \\ (3)(2)+(6)(-1) & (3)(6)+(6)(-3) \end{bmatrix}} = \overset{2\times 2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}.$$

Remark 3.8: Let $\boldsymbol{D_{m\times p}}$ and $\boldsymbol{C_{p\times n}}$ be two matrices.
1- From above examples (i) and (ii), if $\boldsymbol{DC}$ is defined not necessary $\boldsymbol{CD}$ is defined.
2- From above examples (iii) and (iv) it is clear that not necessary $\boldsymbol{DC} = \boldsymbol{CD}$, that is, matrix multiplication is not commutative.

Theorem 3.9: Let $A = [a_{ij}]_{n\times p}$ and $B = [b_{ij}]_{p\times m}$.

(i) The ith row of the matrix product AB is equal to the matrix product of A_iB, where A_i is the row of A.

(ii) The jth column of the matrix product AB is equal to the matrix product of AB_j, where B_j is the column of B.

(iii) If A has a row of zeros, then AB has a row of zeros.

(iv) If B has a column of zeros, then AB has a column of zeros.

Definition3.10: Let $A = [a_{ij}]$ be an $n \times n$ matrix, then

(i) A is called idempotent if $A = A^2$.

(ii) A is called nilpotent if $A^k = 0$ for some integer if k.

Example 3.11:

$A = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}_{3\times 3}$ is idempotent and $B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$ is nilpotent since $B^3 = 0$.

Theorem 3.12: (Properties of Matrix Addition)
(1) $A+B=B+A$. **(Commutative Property for Addition).**
(2) $A+(B+C)=(A+B)+C$. **(Associative Property for Addition).**
(3) $A+0=A$.
(4) $A+(-A)=(-A)+A=0$. **(Additive Inverse).**

Theorem 3.15: (Properties of Scalar Multiplication)
Let k and h be two real numbers and let A and B be two matrices of dimension $m \times n$. Then
(1) $k(hA)=(kh)A$.
(2) $(k+h)A=kA+hA$.
(3) $k(A+B)=kA+kB$.
(4) $0 \cdot A=0$.
(5) $\underbrace{A+A+\cdots+A}_{n-times}=n \cdot A$.

Theorem 3.16: (Properties of Matrix Multiplication)
(1) if A, B and C are of the appropriate sizes, then

$$A(BC)=(AB)C$$ **. (Associative Property)**

(2) if A, B and C are of the appropriate sizes, then

$$A(B+C)=AB+AC$$ **. (Distributive Property)**

(3) if A, B and C are of the appropriate sizes, then

$$(A+B)C=AC+BC.$$

(4) if $A_{m\times n}$ is a matrix, then

$$I_m \cdot A=A \cdot I_n=A.$$

Proof: (1) Let $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{n\times p}$ and $C=[c_{ij}]_{p\times q}$
$AB=[u_{ij}]_{m\times p}$, $u_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$.
$BC=[v_{ij}]_{n\times q}$, $v_{ij}=\sum_{r=1}^{p} b_{ir}c_{rj}$.
Now

$$(i,j)-\text{element of }(AB)C=\sum_{r=1}^{p} u_{ir}c_{rj}=\sum_{r=1}^{p}\left(\sum_{k=1}^{n} a_{ik}b_{kr}\right)c_{rj}$$

$$=\sum_{r=1}^{p}(a_{i1}b_{1r}+a_{i2}b_{2r}+\cdots+a_{in}b_{nr})\,c_{rj}$$

$$= (a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{in}b_{n1})c_{1j} + (a_{i1}b_{12} + a_{i2}b_{22} + \cdots + a_{in}b_{n2})c_{2j} + \cdots + \left(a_{i1}b_{1p} + a_{i2}b_{2p} + \cdots + a_{in}b_{np}\right)c_{pj}$$

$$= a_{i1}\left(b_{11}c_{1j} + b_{12}c_{2j} + \cdots + b_{1p}c_{pj}\right) + a_{i2}\left(b_{21}c_{1j} + b_{22}c_{2j} + \cdots + b_{2p}c_{pj}\right) + a_{in}\left(b_{n1}c_{1j} + b_{n2}c_{2j} + \cdots + b_{np}c_{pj}\right)$$

$$= a_{i1}\left(\sum_{r=1}^{p} b_{1r}c_{rj}\right) + a_{i2}\left(\sum_{r=1}^{p} b_{2r}c_{rj}\right) + \cdots + a_{in}\left(\sum_{r=1}^{p} b_{nr}c_{rj}\right)$$

$$= \sum_{k=1}^{n} a_{ik}\left(\sum_{r=1}^{p} b_{kr}c_{rj}\right) = \sum_{k=1}^{n} a_{ik}v_{kj} = (i,j) - \text{element of } A(BC).$$

(2), (3) and (4) Exercise.

Definition 3.17: Suppose A is a square matrix. If p is a positive integer, then

$$A^p = \underbrace{A \cdot A \cdots A}_{p\ times}.$$

If A is $n \times n$, we define $A^0 = I_n$.

Theorem 3.18: Let p, q are nonnegative integers and A, B are square matrix. Then

(1) $A^p A^q = A^{p+q}$.

(2) $(A^p)^q = (A^q)^p = A^{pq}$.

(3) It is not necessary that

$$(AB)^p = A^p B^p.$$

However, if $AB = BA$, then this rule does hold.

Example 3.19: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$.

$AB = \begin{bmatrix} 8 & 8 \\ 10 & 7 \end{bmatrix}$, $(AB)^2 = \begin{bmatrix} 144 & 120 \\ 150 & 129 \end{bmatrix}$,

$A^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, $B^2 = \begin{bmatrix} 20 & 14 \\ 14 & 13 \end{bmatrix}$, $A^2B^2 = \begin{bmatrix} 156 & 122 \\ 150 & 121 \end{bmatrix}$,

but $(AB)^2 \neq A^2B^2$.

Remark 3.20:

(1) The cancellation law does not hold for matrices as the following example shows.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$. Then

$$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix}.$$

But $B \neq C$.
(2) AB may be zero with neither A nor B equal to zero; that is, if A and B are two nonzero matrices, it is not necessary $AB \neq \mathbf{0}$. That is, the zero property does not hold for matrix multiplication as the following example shows.
Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} -8 & -16 \\ 4 & 8 \end{bmatrix}$.

Theorem 3.21:
(i) (Triangular Matrices)
(1) The sum and product of upper triangular matrices is upper triangular.
(2) The sum and product of lower triangular matrices is lower triangular.
Proof:
(1) Let $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$ and $a_{ij} = b_{ij} = 0$ if $i > j$.
Sum:
Let $A + B = [c_{ij}]_{n\times n}$ where $c_{ij} = a_{ij} + b_{ij}$, to prove $c_{ij} = 0$ if $i > j$. Since $a_{ij} + b_{ij} = 0$ if $i > j$, thus $c_{ij} = 0$. That is, $A + B$ is upper triangular.
Product:
Let $A \cdot B = [c_{ij}]_{n\times n}$ where $c_{ij} = \sum_{t=1}^{n} a_{it} b_{tj}$. T.P. $c_{ij} = 0$ if $i > j$.
If $i > j > t \Rightarrow i > t$, then $a_{it} = 0$.
If $t > i > j \Rightarrow t > j$, then $b_{tj} = 0$.
If $i > t > j \Rightarrow i > t$ and $t > j$ then $a_{it} = b_{tj} = 0$.
Therefore, $c_{ij} = 0$ if $i > j$. That is, $A \cdot B$ is upper triangular.
(ii) (Transpose)
Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$
(1) ${A^T}^T = A$. **(2)** $(AB)^T = B^T A^T$.
(3) If $AA^T = 0$ then $A = 0$, where all entries of A are real numbers.
(4) $(kA)^T = kA^T$, where k is any nonzero real number.
Proof:
(2) Let $A^T = [c_{ij}] = [a_{ji}]$ and $B^T = [d_{ij}] = [b_{ji}]$. Then
$(i,j) -$ element of $(AB)^T = (j,i) -$ element of AB
$= \sum_{k=1}^{p} a_{jk} b_{ki} = \sum_{k=1}^{p} c_{kj} d_{ik} = \sum_{k=1}^{p} d_{ik} c_{kj} = (i,j) -$ element of $B^T A^T$.
(3) Let $A^T = [c_{ij}]$.
If $AA^T = 0$, then $c_{ij} = 0$ for all i, j.$\Rightarrow\Rightarrow$ $c_{ii} = 0$ for all i.
$\Rightarrow\Rightarrow$ $c_{ii} = \sum_{k=1}^{n} a_{ik} c_{ki} = 0$. But $c_{ki} = a_{ik}$. Thus,
$c_{ii} = \sum_{k=1}^{n} a_{ik} a_{ik} = \sum_{k=1}^{n} (a_{ik})^2 = 0 \Leftrightarrow a_{ik} = 0$ for all i and k. Therefore, $A = 0$.
(1), (4) Exercise.

(ii) (Symmetric and Skew Symmetric)
Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$
(1) AA^T and A^TA are symmetric
(2) $A + A^T$ is symmetric.
(3) $A - A^T$ is skew symmetric.
(4) If A is skew symmetric, then A^2 is symmetric.
If A and B are symmetric, then
(5) A is symmetric iff A^T is symmetric.
(6) $A + B$ is symmetric.
(7) AB is symmetric iff $AB = BA$.
(iii) (Conjugate and Transpose Conjugate)
Let $A = [a_{ij}]_{n\times p}$ and $B = [b_{ij}]_{p\times m}$. Then
(1) $\overline{\overline{A}} = A$. **(2)** $\overline{AB} = \overline{A}\,\overline{B}$.
(3) $\overline{kA} = k\overline{A}$, where k is any nonzero real number.
(4) $\overline{A + B} = \overline{A} + \overline{B}$.
If A^θ and B^θ are the transposed conjugate of A and B respectively, then
(5) $(A^\theta)^\theta = A$. **(6)** $(A + B)^\theta = A^\theta + B^\theta$.
(7) $(kA)^\theta = \overline{k}A^\theta$, where k is any nonzero complex number.
(8) $(AB)^\theta = B^\theta A^\theta$.
If $n = p$, then
(9) $A + A^\theta$ is Hermitian matrix.
(10) $A - A^\theta$ is skew Hermitian matrix.
(iv) (Trace)
Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$, then
(1) $\mathrm{Tr}(kA) = k\mathrm{Tr}(A)$, where k is any nonzero real number.
(2) $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$.
(3) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.
(4) $\mathrm{Tr}(A^T) = \mathrm{Tr}(A)$.
Proof:
Let $(i,j) -$ element of $AB = c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ and
$(i,j) -$ element of $BA = d_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$.
$\Longrightarrow c_{ii} = \sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n} b_{ki}a_{ik}$ belong to the main diagonal of AB.
$\Longrightarrow \mathrm{Tr}(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{n}\sum_{i=1}^{n} \underbrace{b_{ki}}_{\in B}\, \underbrace{a_{ik}}_{\in A}$
$= \sum_{k=1}^{n} d_{kk} = \mathrm{Tr}(BA)$.

Echelon Form of a Matrix

Definition 3.22: An $m \times n$ matrix A is said to be in **reduced row echelon form** if it satisfies the
following properties:

(i) All zero rows, if there are any, appear at the bottom of the matrix.
(ii) The first nonzero entry from the left of a nonzero row is a **l**. This entry is called a **leading** one (**leading entry**) of its row.
(iii) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.
(iv) If a column contains a leading one, then all other entries in that column are zero .

An $m \times n$ matrix satisfying properties (i), (ii) and (iii) is said to be **in row echelon form.**

Definition 3.23: The first column with a nonzero entry (counting from left to right) is called the **pivot column** and the first nonzero entry in the pivot column (counting from top to bottom) is called the **pivot**.

Remark: A matrix in reduce row echelon form (row echelon form) might not have any rows that consist entirely of zeros.

Example 3.24:

(i) The following are matrices in reduced row echelon form.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(ii) The matrices that follow arc not in reduced row echelon form.

$$D = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

$$F = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We shall now show that every matrix can be put into row echelon form, or into reduced row echelon form. by means of certain row operations.

(iii)

$$A = \begin{bmatrix} 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}.$$

Pivot column **Pivot**

Definition 3.25: An elementary row (column) operation on a matrix *A* is anyone of the following operations:

(1) Interchange any two rows (columns).

$$\mathbf{r}_i \leftrightarrow \mathbf{r}_j \quad (\mathbf{c}_i \leftrightarrow \mathbf{c}_j).$$

(2) Multiply a row (column) by a nonzero number.

$$k\mathbf{r}_i \to \mathbf{r}_i \quad (k\mathbf{c}_i \to \mathbf{c}_i).$$

(3) Add a multiple of one row (column) to another.

$$k\mathbf{r}_i + \mathbf{r}_j \to \mathbf{r}_j \quad (k\mathbf{c}_i + \mathbf{c}_j \to \mathbf{c}_j).$$

Example 3.25: Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}.$$

Interchanging rows 1 and 3 of *A*, we obtain

$$B = A_{\mathbf{r}_1 \leftrightarrow \mathbf{r}_3} = \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Multiplying the third row of *A* by $\frac{1}{3}$, we obtain

$$C = A_{\frac{1}{3}r_3 \to r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}.$$

Adding (-2) times row 2 of A to row 3 of A. we obtain

$$D = A_{-2r_2 + r_3 \to r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}.$$

Definition 3.26: An $m \times n$ matrix B is said to **be row (column) equivalent** to an $m \times n$ matrix A if B can be produced by applying a finite sequence of elementary row (column) operations to A.

Example 3.27: Let

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -2 & 2 & 3 \end{bmatrix}.$$

If we add 2 times row 3 of A to its second row, we obtain

$$B = A_{2r_3 + r_2 \to r_2} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & -3 & 7 & 8 \\ 1 & -2 & 2 & 3 \end{bmatrix},$$

so B is row equivalent to A.

Theorem 3.28:

(1) Every matrix is row(column) equivalent to itself.

(2) If A is row(column) equivalent to B, then B is row(column) equivalent to A.

(3) If A is row(column) equivalent to B and is B row(column) equivalent to C, then A is row(column) equivalent to C.

(4) Every nonzero $m \times n$ matrix A is row (column) equivalent to a to a unique matrix in row (column) echelon form.

Remark 3.29: It should be noted that a row echelon form of a matrix is not unique.

Example 3.30:
(1) Let

$$A=\begin{bmatrix}0&2&3&-4&1\\0&0&2&3&4\\2&2&-5&2&4\\2&0&-6&9&7\end{bmatrix}.$$

Solution:
Step 1: Identify the pivot column and the pivot.
Step 2: Interchange the first row if necessary with the row where the pivot occurs.

$$A=\begin{bmatrix}0&2&3&-4&1\\0&0&2&3&4\\2&2&-5&2&4\\2&0&-6&9&7\end{bmatrix}.$$

Pivot column **Pivot**

$$\begin{bmatrix}0&2&3&-4&1\\0&0&2&3&4\\2&2&-5&2&4\\2&0&-6&9&7\end{bmatrix}\xrightarrow{r_1\leftrightarrow r_3}\begin{bmatrix}2&2&-5&2&4\\0&0&2&3&4\\0&2&3&-4&1\\2&0&-6&9&7\end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}r_1\rightarrow r_1}\begin{bmatrix}1&1&-\frac{5}{2}&1&2\\0&0&2&3&4\\0&2&3&-4&1\\2&0&-6&9&7\end{bmatrix}$$

$$\xrightarrow{-2r_1+r_4\rightarrow r_4}\begin{bmatrix}1&1&-\frac{5}{2}&1&2\\0&0&2&3&4\\0&2&3&-4&1\\0&-2&-1&7&3\end{bmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}r_2 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}r_3 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{-2r_3 + r_4 \leftrightarrow r_4} \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix

$$H = \begin{bmatrix} 1 & 1 & -\frac{5}{2} & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row echelon form and is row equivalent to A.

$$\xrightarrow[\substack{-\frac{3}{2}\mathbf{r}_3 + \mathbf{r}_2 \to \mathbf{r}_2 \\ \frac{5}{2}\mathbf{r}_3 + \mathbf{r}_1 \to \mathbf{r}_1 \\ -1\mathbf{r}_2 + \mathbf{r}_1 \to \mathbf{r}_1}]{} \begin{bmatrix} 1 & 0 & 0 & 9 & \frac{19}{2} \\ 0 & 1 & 0 & -\frac{17}{4} & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is in reduced row echelon form and is row equivalent to A.

(2) Let

$$A=\begin{bmatrix}1 & -1 & 1\\ 0 & 2 & -1\\ 2 & 3 & 0\end{bmatrix}$$

$$\xrightarrow{-2r_2+r_3\leftrightarrow r_3}\begin{bmatrix}1 & -1 & 1\\ 0 & 2 & -1\\ 0 & 5 & -2\end{bmatrix}\xrightarrow{\frac{1}{2}\,r_2\leftrightarrow r_2}\begin{bmatrix}1 & -1 & 1\\ 0 & 1 & \frac{-1}{2}\\ 0 & 5 & -2\end{bmatrix}$$

$$\xrightarrow{-5r_2+r_3\leftrightarrow r_3}\begin{bmatrix}1 & -1 & 1\\ 0 & 1 & \frac{-1}{2}\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\xrightarrow{2\,r_3\leftrightarrow r_3}\begin{bmatrix}1 & -1 & 1\\ 0 & 1 & \frac{-1}{2}\\ 0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow[-r_3+r_1\leftrightarrow r_1]{-r_3+r_1\leftrightarrow r_1}\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\xrightarrow{r_2+r_3\leftrightarrow r_1}H=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

The matrix H is in reduce row echelon form of A.

Chapter Four
The Inverse of a Matrix

Definition 4.1: If A is a square matrix of order n and if there exists a matrix A^{-1} (read "A inverse") such that

$$AA^{-1} = A^{-1}A = I_n$$

then

A^{-1} is called the **multiplicative inverse of** A or, more simply, the **inverse of** A**.** If no such matrix exists, then A is said to be a **singular matrix(or noninvertible).**

Example 4.2:

(1) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ such that

$$AA^{-1} = A^{-1}A = I_2.$$

We can write

$$\begin{array}{ccccc} A & A^{-1} & & I \\ \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} & \begin{bmatrix} a & c \\ b & d \end{bmatrix} & = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} (2a+3b) & (2c+3d) \\ (a+2b) & (c+2d) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2a + 3b = 1 \qquad\qquad 2c + 3d = 0$$
$$a + 2b = 0 \qquad\qquad c + 2d = 1$$

Using substitution method or elimination method to solve the systems.

$a = 2, \quad b = -1$ $\qquad\qquad$ $c = -3, \quad d = 2$

Therefore,

$$A^{-1} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

(2) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, find A^{-1} if exist.

Solution:

Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $AA^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\Rightarrow AA^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$a + 2c = 1 - \dashrightarrow E_1$$
$$2a + 4c = 0 - \dashrightarrow E_2$$
$$b + 2d = 0 - \dashrightarrow E_3$$
$$2b + 4d = 1 - \dashrightarrow E_4$$

$-2E_1 + E_2 \longrightarrow 0 = -2 \quad C!$. So, the linear systems have no solution. Therefore A has no inverse. That is, A is singular.

(3) Find the inverse, if it exists, of

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

Solution:

$$\overset{A}{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Equating corresponding terms, we see that this is true only if

$$\begin{array}{lll} a - b + c = 1 & d - e + f = 0 & g - h + i = 0 \\ \quad 2b - c = 0 & \quad 2e - f = 1 & \quad 2h - i = 0 \\ 2a + 3b \quad = 0 & 2d + 3e \quad = 0 & 2g + 3h \quad = 1 \end{array}$$

Use substitution or elimination methods to solve these systems.

$$a = 3, b = -2, c = -4$$
$$d = 3, e = -2, f = -5$$
$$g = -1, h = 1, i = 2.$$

Therefore,

$$A^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

4.3: Steps for Finding the Inverse of a Matrix of Dimension $n \times n$

STEP 1: Form the matrix

STEP 2: Using row operations, write$[A|I_n]$ in reduced row echelon form.

STEP 3: If the resulting matrix is of the form $[I_n|B]$ that is, if the identity matrix appears on the left side of the bar, then B is the inverse of A. Otherwise, A has no inverse.

Example 4.4:

(1) Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution:

STEP 1 Since A is of dimension 3×3, use the identity matrix I_3. The matrix $[A|I_3]$ is

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

STEP 2 Proceed to obtain the reduced row echelon form of this matrix:

Use $\begin{matrix} R_2 = -2r_1 + r_2 \\ R_3 = -1r_1 + r_3 \end{matrix}$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -4 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$

Use $R_2 = -1r_2$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$

Use $\begin{matrix} R_1 = -1r_2 + r_1 \\ R_3 = -1r_2 + r_3 \end{matrix}$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 4 & 2 & -1 & 0 \\ 0 & 0 & -4 & -3 & 1 & 1 \end{array}\right]$

Use $R_3 = -\frac{1}{4}r_3$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 4 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$

Use $\begin{aligned} R_1 &= 2r_3 + r_1 \\ R_2 &= -4r_3 + r_2 \end{aligned}$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$

The matrix $[A|I_3]$ is in reduce row echelon form.
STEP 3: Since the identity matrix I_3 appears on the left side, the matrix appearing on the right is the inverse. That is,

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(2) Show that the matrix given below has no inverse.

$$A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}.$$

Solution: Set up the matrix

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 6 & 4 & 0 & 1 \end{array}\right]$$

Use $R_1 = \frac{1}{3}r_1$ to obtain $\left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 6 & 4 & 0 & 1 \end{array}\right]$

Use $R_2 = -6r_1 + r_2$ to obtain $\left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$

The 0s in row 2 tell us we cannot get the identity matrix. This, in turn, tells us the original matrix has no inverse.

Theorem 4.5:

(1) If a matrix has an inverse, then the inverse is unique.

(2) The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular $\Leftrightarrow ad - bc \neq 0$ and

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

(3) An $n \times n$ matrix is nonsingular $\Leftrightarrow$ it is row equivalent to I_n.

(4) An $n \times n$ matrix A is singular $\Leftrightarrow A$ is row equivalent to a matrix B that has a row of zeros. (That is, the reduced row echelon form of A has a row of zeros .)

(5) If A is a nonsingular matrix, then A^{-1} is nonsingular and ${A^{-1}}^{-1} = A$.

(6) If A and B are nonsingular matrices, then AB is nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(7) If A is a nonsingular matrix, then $(A^{-1})^T = (A^T)^{-1}$.

(8) If $A_1, A_2, \cdots, A_r$ are $n \times n$ nonsingular matrices, then $A_1A_2 \cdot \cdots \cdot A_r$ is nonsingular and $(A_1A_2 \cdot \cdots \cdot A_r)^{-1} = {A_r}^{-1}{A_{r-1}}^{-1} \cdot \cdots \cdot {A_1}^{-1}$.

Proof:

(1) Let B and C be inverse matrices of a matrix A. Then $AB = BA = I$ and $AC = CA = I$. To prove $B = C$.

$$B = BI = B(AC) = (BA)C = IC = C \Rightarrow B = C.$$

(2) Check.

(3) and **(4)** without prove.

(5) Check.

(6) Since

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

Therefore, $(AB)^{-1} = B^{-1}A^{-1}$.

(7) We have

$$AA^{-1} = A^{-1}A = I_n \Rightarrow (AA^{-1})^T = (A^{-1}A)^T = A^T(A^{-1})^T = {I_n}^T = I_n.$$

Then, $(A^{-1})^TA^T = (AA^{-1})^T = (A^{-1}A)^T = A^T(A^{-1})^T = {I_n}^T = I_n$.

(8) Without prove.

Chapter Five
Determinants

Definition 5.1: (First- and Second-Order Determinants)
For any square matrix A, the **determinant** of A is a real number denoted by **det(A)** or $|A|$. If A is a square matrix of order n, then det(A) is called a **determinant of order *n*.** If $A = [a_{11}]$ is a square matrix of order 1, then

$$\det(A) = a_{11}$$

is a **first-order determinant.**

Given a second-order square matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the **second-order determinant** of A is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}\,.$$

Example 5.2:

Find $\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix}$.

Solution:

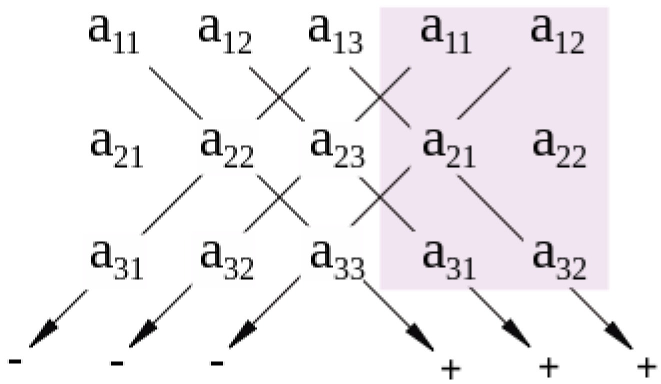
$$\det(A) = \begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix}$$
$$= (-1)(-4) - (-3)(2)$$
$$= 4 - (-6)$$
$$= 10$$

Evaluating Third-Order Determinants

Definition 5.3: Given the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, **the third-order determinant** of A is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}\,.$$

We can also obtain $|A|$ using the following diagram(Sarrus diagram).

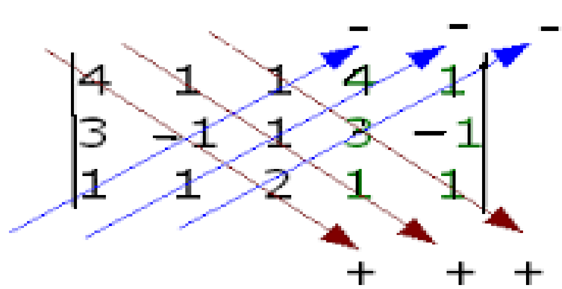


$|A|$=sum of the product of the entries on each line.

Example: Find $|A|$ where $A = \begin{bmatrix} 4 & 1 & 1 \\ 3 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution: By using Sarrus diagram we get the following.

$$|A|$$



$$\begin{aligned} |A| &= (4 \cdot -1 \cdot 2) + (1 \cdot 1 \cdot 1) + (1 \cdot 3 \cdot 1) - (1 \cdot -1 \cdot 1) - (1 \cdot 1 \cdot 4) - (2 \cdot 3 \cdot 1) \\ &= -8 + 1 + 3 + 1 - 4 - 6 = -13. \end{aligned}$$

Definition 5.4: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let M_{ij} be an $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of A. The determinant $\det(M_{ij})$ is called the **minor** of a_{ij}.

Definition 5.5: Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **cofactor** A_{ij} of a_{ij} is defined as

$$\begin{aligned} A_{ij} &= (-1)^{i+j} \det(M_{ij}) \\ &= (-1)^{i+j} (\text{ minor of } a_{ij}). \end{aligned}$$

The **minor of an element** in a third-order determinant is a second-order determinant obtained by deleting the row and column that contains the element.

Deletions are usually done mentally.

$$\text{Minor of } a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$$

$$\text{Minor of } a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{23} - a_{21}a_{13}$$

Example 5.6: Find the cofactors of -2 and 5

$$A = \begin{bmatrix} -2 & 0 & 3 \\ 1 & -6 & 5 \\ -1 & 2 & 0 \end{bmatrix}.$$

Solution:

$$\text{Cofactor of } -2 = (-1)^{1+1}\begin{vmatrix} -6 & 5 \\ 2 & 0 \end{vmatrix} = \begin{vmatrix} -6 & 5 \\ 2 & 0 \end{vmatrix}$$

$$= (-6)(0) - (2)(5) = -10$$

$$\text{Cofactor of } 5 = (-1)^{2+3}\begin{vmatrix} -2 & 0 \\ -1 & 2 \end{vmatrix} = -\begin{vmatrix} -2 & 0 \\ -1 & 2 \end{vmatrix}$$

$$= -[(-2)(2) - (-1)(0)] = 4$$

If we think of the sign $(-1)^{i+j}$ as being located in position (i, j) of an $n \times n$ matrix, then the signs form a checkerboard pattern that has a + in the (1,1) position. The patterns for $n = 3$ and $n = 4$ are as follows:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$n = 3 \qquad\qquad n = 4$$

Theorem 5.7:

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$
$$[\text{expansion of } \det(A) \text{ along the } i\text{th row}]$$

and

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$
$$[\text{expansion of } \det(A) \text{ along the } j\text{th column}].$$

Example 5.8: (1) Evaluate

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix}$$

Solution:

We can choose any row or column to expand along. We will choose the first row because of the zero: we won't need to find that cofactor because it will be multiplied by zero.

$$\begin{aligned}
\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix} &= a_{11}\begin{pmatrix}\text{Cofactor} \\ \text{of } a_{11}\end{pmatrix} + a_{12}\begin{pmatrix}\text{Cofactor} \\ \text{of } a_{12}\end{pmatrix} + a_{13}\begin{pmatrix}\text{Cofactor} \\ \text{of } a_{13}\end{pmatrix} \\
&= 2\left[(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix}\right] + (-2)\left[(-1)^{1+2}\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix}\right] + 0 \\
&= (2)(1)[(1)(-1) - (-3)(2)] + (-2)(-1)[(-3)(-1) - (1)(2)] \\
&= (2)(5) + (2)(1) = 12
\end{aligned}$$

(2) Evaluate

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix} = (-1)^{3+1}(3)\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} + (-1)^{3+2}(0)\begin{vmatrix} 1 & -3 & 4 \\ -4 & 1 & 3 \\ 2 & -2 & 3 \end{vmatrix}$$

$$+ (-1)^{3+3}(0)\begin{vmatrix} 1 & 2 & 4 \\ -4 & 2 & 3 \\ 2 & 0 & 3 \end{vmatrix} + (-1)^{3+4}(-3)\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

$$=(3)\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} +(3)\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

$$= (3)\left(2\begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} - 2\begin{vmatrix} -3 & 4 \\ -2 & 3 \end{vmatrix}\right) + (3)\left((-2)\begin{vmatrix} -4 & 1 \\ 2 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ 2 & -2 \end{vmatrix}\right)$$

$$= (3)(2)(3+6) - (3)(2)(-9+8) + (3)(-2)(8-2) + (3)(2)(-2+6)$$

$$= 54 + 6 - 36 + 24 = 48.$$

(3) Given the fourth order determinant

$$\begin{vmatrix} 0 & -1 & 0 & 2 \\ -5 & -6 & 0 & -3 \\ 4 & 5 & -2 & 6 \\ 0 & 3 & 0 & -4 \end{vmatrix}.$$

(i) Find the minor of the element $a_{33} = -2$.

(ii) Find the cofactor of a_{33}.

(iii) Find the value of the fourth order determinate.

Solution:

(i) Minor of $-2 = \begin{vmatrix} 0 & -1 & 2 \\ -5 & -6 & -3 \\ 0 & 3 & -4 \end{vmatrix} = (-5)(-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} = 5(4-6) = -10.$

(ii) Cofactor of $-2 = (-1)^{3+3}(\text{minor of} -2) = (\text{minor of} -2) = -10.$

(iii) $\begin{vmatrix} 0 & -1 & 0 & 2 \\ -5 & -6 & 0 & -3 \\ 4 & 5 & -2 & 6 \\ 0 & 3 & 0 & -4 \end{vmatrix} = (-2)(\text{cofactor of} - 2) = (-2)(-10) = 20.$

Theorem 5.9: (Determinant properties)

(1) If a matrix B results from a matrix A by interchanging two rows (columns) of A, then $|B| = -|A|$.

(2) If two rows (column) of A are equal, then $|A| = 0$.

(3) If two rows (column) of A consists entirely of zeros, then $|A| = 0$.

(4) If a matrix B is obtained from a matrix A by multiplying a row (column) of A by a real number $c \neq 0$, then $|B| = c|A|$.

(5) If a matrix $B = [b_{ij}]$ is obtained from a matrix $A = [a_{ij}]$ by adding to each element of the rth row (column) of A a nonzero constant c times the corresponding element of the sth row (column) $r \neq s$ of A, then $|B| = |A|$.

$$\begin{vmatrix} a & b & \alpha a + x \\ c & d & \alpha c + y \\ e & f & \alpha e + z \end{vmatrix} = \begin{vmatrix} a & b & x \\ c & d & y \\ e & f & z \end{vmatrix}.$$

(6) If each element of a row (column) of a determinant is the sum of two numbers, then, the determinant can be expressed as the sum of two determinants.

$$\begin{vmatrix} a & b & \alpha + x \\ c & d & \beta + y \\ e & f & \gamma + z \end{vmatrix} = \begin{vmatrix} a & b & \alpha \\ c & d & \beta \\ e & f & \gamma \end{vmatrix} + \begin{vmatrix} a & b & x \\ c & d & y \\ e & f & z \end{vmatrix}.$$

(7) The determinant of a matrix and its transpose are equal; that is, $|A| = |A^T|$.

(8) If a matrix $A = [a_{ij}]_{n\times n}$ is upper (lower) triangular, then

$$A = a_{11} \cdot a_{22} \cdot \cdots \cdot a_{nn}$$

= Product of the elements on the main diagonal.

(9) $|AB| = |A||B|$.

(10) If A is a nonsingular matrix, then $|A| \neq 0$ and $|A^{-1}| = \frac{1}{|A|}$.

Proof:

Prove (1),(2),(3) and (4) using matrices of dimension two.

(5) $\begin{vmatrix} a & b & \alpha a + x \\ c & d & \alpha c + y \\ e & f & \alpha e + z \end{vmatrix} = a\begin{vmatrix} d & \alpha c + y \\ f & \alpha e + z \end{vmatrix} - b\begin{vmatrix} c & \alpha c + y \\ e & \alpha e + z \end{vmatrix} + (\alpha a + x)\begin{vmatrix} c & d \\ e & f \end{vmatrix}$

$= a(d\alpha e + dz - f\alpha c - fy) - b(c\alpha e + cz - e\alpha c - ey) + (\alpha a + x)(cf - ed).$

$= ad\alpha e + adz - af\alpha c - afy - bc\alpha e - bcz + be\alpha c + bey + \alpha acf - \alpha aed + xcf - xed.$

$= adz - afy - bcz + bey + xcf - xed.$
$= a(dz - fy) - b(cz - ey) + x(cf - ed).$
$= a\begin{vmatrix} d & y \\ f & z \end{vmatrix} - b\begin{vmatrix} c & y \\ e & z \end{vmatrix} + x\begin{vmatrix} c & d \\ e & f \end{vmatrix} = \begin{vmatrix} a & b & x \\ c & d & y \\ e & f & z \end{vmatrix}.$

(6) Check.
(7) Prove it using matrix of dimension 2.
(8) Prove it using matrix of dimension 3.
(9) Without prove.
(10) If A is nonsingular matrix, then there exist A^{-1} such that

$$AA^{-1} = A^{-1}A = I_n.$$

$\Longrightarrow |AA^{-1}| = |I_n|$. But $|I_n| = 1$. Also, $\Longrightarrow |AA^{-1}| = |A||A^{-1}| = 1$

$$\Longrightarrow |A||A^{-1}| = 1 \Longrightarrow |A^{-1}| = \frac{1}{|A|}$$

Example 5.10:

(1) Let $A = \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix}$, $A^T = \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix}$. $\Longrightarrow |A| = (2 \cdot 0) - (2 \cdot 1) = -2.$
$\Longrightarrow |A^T| = (2 \cdot 0) - (1 \cdot 2) = -2.$

(2) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$. Find $|A|$ using cofactor expansion.

Solution:

$|A| = 1\begin{vmatrix} 5 & 6 \\ 0 & 0 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 0 & 0 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 0 & 0 \end{vmatrix}$=(1)(0) – (2)(0)+(3)(0)=0.

(3) Let $A = \begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix}$. Find $|A|$.

Solution:

$$|A| = \begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = \begin{vmatrix} 2 & 2 \cdot 3 \\ 1 & 12 \end{vmatrix} = (2)\begin{vmatrix} 1 & 3 \\ 1 & 3 \cdot 4 \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = (2)(3)(4 - 3)$$
$$= 6.$$

(4) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 8 & 6 \end{bmatrix}$. Find $|A|$ using properties only.

Solution:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 8 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 2 \cdot 4 & 2 \cdot 3 \end{vmatrix} = (2)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 1 & 4 & 3 \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 1 \\ 1 & 4 & 1 \end{vmatrix}$$
$$= (2)(3)(0) = 0.$$

(5) Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$.

$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7.$

Let $A_{c_1 \leftrightarrow c_2} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} = B.$

$|B| = \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} = -3 - 4 = -7.$

(6) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 = 2r_2 + r_1} \begin{bmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix} = B. \Longrightarrow |A| = |B| = 4.$$

(7) Let $A = \begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{bmatrix}$. Use determinant properties only to find $|A|$.

Solution:

$$\begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 2\cdot 2 & 2\cdot 3 \end{vmatrix} \xrightarrow{R_3(2)} (2)\begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_3} (-2)\begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 5 \\ 4 & 3 & 2 \end{vmatrix} \xrightarrow{R_2 = -3r_1 + r_2} (-2)\begin{vmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 4 & 3 & 2 \end{vmatrix}$$

$$\xrightarrow{R_3 = -4r_1 + r_3} (-2)\begin{vmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & -5 & -10 \end{vmatrix} \xrightarrow{R_2(-4)} (-2)(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & -5 & -10 \end{vmatrix}$$

$$\xrightarrow{R_3(-5)} (-2)(-4)(-5)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$\xrightarrow{R_3 = -\frac{1}{2}r_2 + r_3} (-2)(-4)(-5)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{3}{2} \end{vmatrix} = (-2)(-4)(-5)(1)(2)\left(\frac{3}{2}\right)$$

$$= -120.$$

(8) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

$\Longrightarrow AB = \begin{bmatrix} 4 & 3 \\ 10 & 5 \end{bmatrix} \Longrightarrow |A| = -2,\ |B| = 5 \ \Longrightarrow |A||B| = (-2)(5) = -10.$
$\Longrightarrow |AB| = 20 - 30 = -10.$
(9) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
$ad - bc = 4 - 6 = -2 \Longrightarrow A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}, |A| = -2 \Longrightarrow |A^{-1}| = \frac{\mathbf{-1}}{\mathbf{2}} = \frac{\mathbf{1}}{|\boldsymbol{A}|}$**.**
Definition 5.11: Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $n \times n$ matrix $adj(A)$, called the **adjoint** of A, is the matrix whose (i, j)th entry is the cofactor A_{ji} of a_{ji} . Thus

$$adj(A) = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ & & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}.$$

Example 5.12: Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$. Compute $adj(A)$.
Solution: We first compute the cofactors of A. We have

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} = -18,\ A_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} = 17,\quad A_{13} = (-1)^{1+3}\begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} = -6,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6,\ A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -10,\quad A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -2,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} = -10,\ A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = -1,\quad A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ 5 & 6 \end{vmatrix} = 28.$$

Then

$$adj(A) = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}.$$

Theorem 5.13: If A is an $n \times n$ matrix, then

(1) $a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = \det(A)$ if $i = k$,
$= 0$ if $i \neq k$

$a_{1j}A_{1k} + a_{2j}A_{2k} + \cdots + a_{nj}A_{nk} = \det(A)$ if $j = k$,
$= 0$ if $j \neq k$

(2) $Aadj(A) = adj(A)A = |A|I_n$.

Proof:

(1) without prove.

(2) We have

$$A(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{j1} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{j2} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{jn} & \cdots & A_{nn} \end{bmatrix}.$$

From (1) we have that the (i, j)th element in the product matrix $Aadj(A)$

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \det(A) \text{ if , if } i = j,$$
$$= 0 \qquad \text{if } i \neq j.$$

This means that

$$A(\text{adj } A) = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \det(A) \end{bmatrix} = \det(A)I_n.$$

Thus, det(A)

Also, from (1), the (i, j)th element in the product matrix $adj(A)A$ is

$A_{1i}a_{1j} + A_{2i}a_{2j} + \cdots + A_{ni}a_{nj} = \det(A)$ if $i = j$,
$= 0$ if $i \neq j$.

Thus, $Aadj(A) = adj(A)A = |A|I_n$.

Example 5.14: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & 5 & -2 \end{bmatrix}$. Then

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = 19, \quad A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = -14, \quad \text{and} \quad A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 3.$$

Now

$$a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} = (4)(19) + (5)(-14) + (-2)(3) = 0.$$

and

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = (1)(19) + (2)(-14) + (3)(3) = 0.$$

$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = (-2)(19) + (3)(-14) + (1)(3) = -77.$

Theorem 5.15: If A is an $n \times n$ nonsingular matrix, then

$$A^{-1} = \frac{1}{\det(A)}(\operatorname{adj} A) = \begin{bmatrix} \frac{A_{11}}{\det(A)} & \frac{A_{21}}{\det(A)} & \cdots & \frac{A_{n1}}{\det(A)} \\ \frac{A_{12}}{\det(A)} & \frac{A_{22}}{\det(A)} & \cdots & \frac{A_{n2}}{\det(A)} \\ \vdots & \vdots & & \vdots \\ \frac{A_{1n}}{\det(A)} & \frac{A_{2n}}{\det(A)} & \cdots & \frac{A_{nn}}{\det(A)} \end{bmatrix}.$$

Proof: Exercise.

Example 5.16: Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$. Compute A^{-1} using adjoint of A.

Solution:

$$adj(A) = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$$

and $|A| = -94$.

Therefore,

$$A^{-1} = \frac{adj\ (A)}{|A|} = \begin{bmatrix} \frac{-18}{-94} & \frac{-6}{-94} & \frac{-10}{-94} \\ \frac{17}{-94} & \frac{-10}{-94} & \frac{-1}{-94} \\ \frac{-6}{-94} & \frac{-2}{-94} & \frac{28}{-94} \end{bmatrix}.$$

Theorem 5.17: A matrix A is nonsingular $\Leftrightarrow$ $|A| \neq 0$.
Proof: Exercise.

Chapter Six
Solution of Linear Systems

6.1: Consider the linear system of m linear equations in n unknown.

$$\begin{matrix} a_{11}x_1 & a_{12}x_2 & \cdots & \cdots & a_{1n}x_n = b_1 \\ a_{21}x_1 & a_{22}x_2 & \cdots & \cdots & a_{2n}x_n = b_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1}x_1 & a_{m2}x_2 & \cdots & \cdots & a_{mn}x_n = b_m \end{matrix} \quad \text{------------>}\ (1)$$

Now define the following matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m\times n}, B = \begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}, \ X = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n\times 1},$$

$$C = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} & b_m \end{array}\right]_{m\times(n+1)} = [A|B].$$

The matrix A is called the **coefficient matrix** of the linear system (1) and the matrix C is called the **augmented matrix** of the linear system (1).
Then the linear system (1) can be written in matrix form as

$$AX = B \quad \text{------- >}\ (2)$$

The linear system as in (2) is called **homogenous system** if $B = 0$; that is

$$AX = 0 \ \text{---------- >}\ (3)$$

The solution $x_1 = x_2 = \cdots x_n = 0$ to the homogenous system (3) is called **trivial solution.**
A solution $x_{1=}s_1, x_{2=}s_2, \cdots x_n = s_n$ to a homogenous system in which not all the $s_i = 0$ is called **nontrivial solution**.

Example 6.2:

(1) Consider the linear system

$$\begin{aligned} 2x + 3y - 4z &= 5 \\ -2x \qquad - z &= 7 \\ 3x + 2y + 2z &= 3 \end{aligned}$$

The augmented matrix of the linear system is

The coefficient matrix of the linear system is $A = \begin{bmatrix} 2 & 3 & -4 \\ -2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix}$ and the augmented matrix is $\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ -2 & 0 & 1 & 7 \\ 3 & 2 & 2 & 3 \end{array}\right]$. Also, $B = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}_{3\times 1}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}$.
$AX = B$.

(2) The matrix $\left[\begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}\middle| \begin{matrix} 0 \\ 1 \\ 1 \end{matrix}\right]$ is the augmented matrix of the following linear system

$$\begin{aligned} x + 3y - z &= 0 \\ 2x \qquad + z &= 1 \\ 3x + y \qquad &= 1 \end{aligned}$$

Theorem 6.3:
(1) Let $AX = B$ and $CX = D$ be two linear systems each of m equations in n unknown. If the augmented matrices $[A|B]$ and $[C|D]$ of these systems are row equivalent, then both linear systems have exactly the same solutions.
(2) A and C are row equivalent $m \times n$ matrices, then the linear systems $AX = 0$ and $CX = 0$ have exactly the same solutions.
Proof: Without prove.
Theorem 6.4:
(1) A homogenous system of m equations in n unknown always has a nontrivial solution if $m < n$.
(2) If A is an $n \times n$ matrix, then the homogenous system

$$AX = 0$$

has a nontrivial solution $\Leftrightarrow$ A is singular $\Leftrightarrow$ $|A| = 0$.

The following theorem summarizes results on homogenous systems and nonsingular matrices.
Theorem 6.5: Let A be an $n \times n$ matrix, then the following are equivalent:
(1) A is nonsingular.
(2) $AX = 0$ has only the trivial solution.
(3) A is row equivalent to the identity matrix I_n.
(4) The linear system $AX = B$ has a unique solution for every $n \times 1$ matrix B.
(5) $|A| \neq 0$.
Proof: Exercise.

Solving Linear System By Gauss-Jordan Reduction

Carl Friedrich Gauss (1777- 1855)
Germany

Wilhelm Jordan (1842-1899)
Germany

6.6: The Gauss-Jordan reduction procedure for solving a linear system $AX = B$ is as follows:
STEP 1: Form the augmented matrix $[A|B]$.
STEP 2: Transform the augmented matrix to the matrix $[C|D]$ in reduce row echelon form by using elementary row operation.
STEP 3: Solve the corresponding equation for the unknown that corresponds to the leading entry of the row.
Example 6.7: Solve by Gauss–Jordan elimination
(1)

$$2x_1 - 2x_2 + x_3 = 3$$
$$3x_1 + x_2 - x_3 = 7$$
$$x_1 - 3x_2 + 2x_3 = 0$$

Solution: Write the augmented matrix and follow the steps indicated at the right to produce a reduced form.

Need a 1 here.

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

Step 1: Choose the leftmost nonzero column and get a 1 at the top.

Need 0's here.

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right] \quad \begin{array}{l} (-3)R_1 + R_2 \to R_2 \\ (-2)R_1 + R_3 \to R_3 \end{array}$$

Step 2: Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

Need a 1 here.

$$\sim\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 0 & 4 & -3 & 3 \end{array}\right] \quad 0.1R_2 \to R_2$$

Step 3: Repeat step 1 with the *submatrix* formed by (mentally) deleting the top (shaded) row.

Need 0's here.

$$\sim\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 4 & -3 & 3 \end{array}\right] \quad \begin{array}{l} 3R_2 + R_1 \to R_1 \\ \\ (-4)R_2 + R_3 \to R_3 \end{array}$$

Step 4: Repeat step 2 with the *entire matrix.*

Need a 1 here.

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & -0.2 & 0.2 \end{array}\right] \quad (-5)R_3 \to R_3$$

Step 3: Repeat step 1 with the *submatrix* formed by (mentally) deleting the top two (shaded) rows.

Need 0's here.

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} 0.1R_3 + R_1 \to R_1 \\ 0.7R_3 + R_2 \to R_2 \end{array}$$

Step 4: Repeat step 2 with the *entire matrix.*

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The matrix is now in reduced form, and we can proceed to solve the corresponding reduced system.

The system which has the above augmented matrix is

$$\begin{aligned} x_1 + 0 + 0 &= 2 \\ 0 + x_2 + 0 &= 0 \\ 0 + 0 \ + x_1 &= -1 \end{aligned}$$

Therefore, $S.S. = \{(2,0,-1)\}$.

(2)

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &= -4 \\ 4x_1 - 8x_2 + 7x_3 &= 2 \\ -2x_1 + 4x_2 - 3x_3 &= 5 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & -4 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right] \xrightarrow[\text{(To get 1 in upper left corner)}]{0.5R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right]$$

$$\xrightarrow[2R_1 + R_3 \to R_3]{(-4)R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -2 & 1 \end{array}\right] \xrightarrow{0.2R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[2R_2 + R_3 \to R_3]{(-0.5)R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

We stop the Gauss–Jordan elimination, even though the matrix is not in reduced form, since the last row produces a contradiction.
The system is inconsistent and has no solution.

(3)

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 3 & 6 & -9 & 15 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right]$$

$$\xrightarrow[2R_1 + R_3 \leftrightarrow R_3]{(-2)R_1 + R_2 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{(-2)R_2 + R_1 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is now in reduced form. Write the corresponding reduced system and solve.

$x_1 + 0 + x_3 = -3 \quad \Rightarrow x_1 = -x_3 - 3$

$0 + x_2 - 2x_3 = 4 \quad \Rightarrow x_2 = 2x_3 + 4$

This dependent system has an infinite number of solutions. We will use a parameter to represent all the solutions.

$$x_1 = -t - 3$$
$$x_2 = 2t + 4$$
$$x_3 = t$$

Where $t \in R$. Therefore, $S.S. = \{(-t-3, 2t+4, t) | t \in R\}$.

(4)

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 - x_5 &= 1 \\ 2x_1 + 4x_2 + 8x_3 + 3x_4 - 4x_5 &= 2 \\ x_1 + 3x_2 + 7x_3 + \quad\quad 3x_5 &= -2 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 2 & 4 & 8 & 3 & -4 & 2 \\ 1 & 3 & 7 & 0 & 3 & -2 \end{array}\right] \xrightarrow[(-1)R_1 + R_3 \leftrightarrow R_3]{(-2)R_1 + R_2 \leftrightarrow R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 3 & -1 & 4 & -3 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccccc|c} 1 & 2 & 4 & 1 & -1 & 1 \\ 0 & 1 & 3 & -1 & 4 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{(-2)R_2 + R_1 \to R_1} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & -9 & 7 \\ 0 & 1 & 3 & -1 & 4 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 + R_2 \leftrightarrow R_2]{(-3)R_3 + R_1 \leftrightarrow R_1} \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -3 & 7 \\ 0 & 1 & 3 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

This matrix is in reduce row echelon form. Write the corresponding reduced system and solve.

$$
\begin{aligned}
x_1 \quad - 2x_3 \quad - 3x_5 &= 7 \\
x_2 + 3x_3 \quad + 2x_5 &= -3 \\
x_4 - 2x_5 &= 0
\end{aligned}
$$

Solve for the leftmost variables x_1, x_2, and x_4 in terms of the remaining variables x_3 and x_5:

$$
\begin{aligned}
x_1 &= 2x_3 + 3x_5 + 7 \\
x_2 &= -3x_3 - 2x_5 - 3 \\
x_4 &= 2x_5
\end{aligned}
$$

If we let $x_3 = s$and $x_5 = t$, then for any real numbers s and t,

$$
\begin{aligned}
x_1 &= 2s + 3t + 7 \\
x_2 &= -3s - 2t - 3 \\
x_3 &= s \\
x_4 &= 2t \\
x_5 &= t
\end{aligned}
$$

$S.S. = \{(2s + 3t + 7, -3s - 2t - 3, s, 2t, t) | s, t \in R\}$.

(4) A chemical manufacturer plans to purchase a fleet of 24 railroad tank cars with a combined carrying capacity of 250,000 gallons. Tank cars with three different carrying capacities are available: 6000 gallons, 8000 gallons, and 18000 gallons. How many of each type of tank car should be purchased?

Solution:

Let

$$
\begin{aligned}
x_1 &= \text{Number of 6,000-gallon tank cars} \\
x_2 &= \text{Number of 8,000-gallon tank cars} \\
x_3 &= \text{Number of 18,000-gallon tank cars}
\end{aligned}
$$

Then

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 24 \qquad \text{Total number of tank cars} \\
6{,}000x_1 + 8{,}000x_2 + 18{,}000x_3 &= 250{,}000 \qquad \text{Total carrying capacity}
\end{aligned}
$$

Now we can form the augmented matrix of the system and solve by using Gauss–Jordan reduction:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 6{,}000 & 8{,}000 & 18{,}000 & 250{,}000 \end{array}\right] \xrightarrow{\frac{1}{1{,}000}R_2 \to R_2 \text{ (simplify } R_2)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 6 & 8 & 18 & 250 \end{array}\right]$$

$$\xrightarrow{(-6)R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 0 & 2 & 12 & 106 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 24 \\ 0 & 1 & 6 & 53 \end{array}\right]$$

$$\xrightarrow{(-1)R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -29 \\ 0 & 1 & 6 & 53 \end{array}\right]$$

The matrix is in reduce row echelon form.

$$\begin{aligned} x_1 \quad - 5x_3 &= -29 \qquad \text{or} \qquad x_1 = 5x_3 - 29 \\ x_2 + 6x_3 &= 53 \qquad\;\; \text{or} \qquad x_2 = -6x_3 + 53 \end{aligned}$$

Let $x_3 = t$. Then for t any real number,

$$\begin{aligned} x_1 &= 5t - 29 \\ x_2 &= -6t + 53 \\ x_3 &= t \end{aligned}$$

is a solution or is it**?** Since the variables in this system represent the number of tank cars purchased, the values ofx_1, x_2, and x_3 must be nonnegative integers. The third equation requires that t must be a nonnegative integer. The first equation requires that $5t - 29 \geq 0$, so t must be at least 6. The middle equation requires that $-6t + 53 \geq 0$, so t can be no larger than 8.

So, 6, 7, and 8 are the only possible values for t. There are three different possible combinations that meet the company's specifications of 24 tank cars with a total carrying capacity of 250,000 gallons, as shown in Table 1:

Table 1

t	6,000-Gallon Tank Cars x_1	8,000-Gallon Tank Cars x_2	18,000-Gallon Tank Cars x_3
6	1	17	6
7	6	11	7
8	11	5	8

The final choice would probably be influenced by other factors. For example, the company might want to minimize the cost of the 24 tank cars.

Solving Linear System By Gaussian Elimination

6.8: The Gauss-Jordan reduction procedure for solving a linear system $AX = B$ is as follows:
STEP 1: Form the augmented matrix $[A|B]$.
STEP 2: Transform the augmented matrix to the matrix $[C|D]$ in reduce row echelon form by using elementary row operation.
STEP 3: Solution of the linear system corresponding to the augmented matrix $[C|D]$ using back substitution.
Example 6.9: Solve the following system by Gaussian elimination.

$$\begin{aligned} 6x - y - z &= 4 && (1) \\ -12x + 2y + 2z &= -8 && (2) \\ 5x + y - z &= 3 && (3) \end{aligned}$$

Solution:

$$\left[\begin{array}{rrr|r} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \xrightarrow[R_1 = -r_3 + r_1]{} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 = 12r_1 + r_2 \\ R_3 = -5r_1 + r_3}]{} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right] \xrightarrow[R_1 = -\frac{1}{22}r_2]{} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array}\right]$$

$$\xrightarrow[R_3 = -11r_2 + r_3]{} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in row echelon form. Because the bottom row consists entirely of **0**s, the system actually consists of only two equations.

$$\begin{aligned} x - 2y &= 1 \\ y - \frac{1}{11}z &= -\frac{2}{11} \end{aligned}$$

From the second equation we get $y = \frac{1}{11}z - \frac{2}{11}$. Then back-substitute this solution for y into the first equation to get

$$x = 2y + 1 = 2\left(\frac{1}{11}z - \frac{2}{11}\right) + 1 = \frac{2}{11}z + \frac{7}{11}$$

The original system is equivalent to the system

$$\begin{aligned} x &= \frac{2}{11}z + \frac{7}{11} \\ y &= \frac{1}{11}z - \frac{2}{11} \end{aligned}$$

where z, the parameter, can be any real number. If we let $z = t$ then

$$S.S. = \left\{\left(\frac{2}{11}t + \frac{7}{11}, \frac{2}{11}t + \frac{2}{11}, t\right) \middle| \, t \in R\right\}.$$

Solving Linear System By Cramer's Rule

6.10: Let $AX = B$ be an $n \times n$ linear system, where $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Then the Cramer's rule is as follows:

If $|A| \neq 0$, then

$$x_i = \frac{|A_i|}{|A|}, i = 1,2,\cdots,n$$

where A_i is the matrix obtained from A by replacing the ith column by B.

If $n = 3$ then Cramer's rule as follows:

Given the system

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= k_1 \\ a_{21}x + a_{22}y + a_{23}z &= k_2 \\ a_{31}x + a_{32}y + a_{33}z &= k_3 \end{aligned} \qquad \text{with} \qquad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad y = \frac{\begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}}{D} \qquad z = \frac{\begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}}{D}$$

Example 6.11: (Solving a Three-Variable System with Cramer's Rule)
Solve using Cramer's rule:

$$\begin{aligned} x + y &= 2 \\ 3y - z &= -4 \\ x + z &= 3 \end{aligned}$$

Solution:

$$|A| = D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 2$$

$$x = \frac{\begin{vmatrix} 2 & 1 & 0 \\ -4 & 3 & -1 \\ 3 & 0 & 1 \end{vmatrix}}{2} = \frac{7}{2}$$

$$y = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 0 & -4 & -1 \\ 1 & 3 & 1 \end{vmatrix}}{2} = -\frac{3}{2}$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & -4 \\ 1 & 0 & 3 \end{vmatrix}}{2} = -\frac{1}{2}.$$

Remark 6.12:

(1) Cramer's rule is only applicable in the case where we have n equations and n unknowns (that is, coefficient matrix is square) and the coefficient matrix is nonsingular.

(2) Cofactor expansion can be used to find determinants of orders higher than **3**, so Cramer's rule can be used for systems with more than three variables.

(3) For large systems ($n > 4$), Cramer's rule becomes computationally inefficient. However, the Gauss-Jordan method, which involves fewer arithmetic operations than Cramer's Rule, is a more practical choice.

Solving Linear System Using Inverses

6.13: In general, any system of n linear equations containing n variables $x_1, x_2, \dots, x_n$ can be written in the form

$$AX = B$$

where A is the $n \times n$ matrix of the coefficients of the variables, B is an $n \times 1$ column matrix whose entries are the numbers appearing to the right of each equal sign in the system, and X is an $n \times 1$ column matrix containing the n variables.

To find X, start with the matrix equation $AX = B$ and use properties of matrices. Assume that the $n \times n$ matrix A has an inverse A^{-1}; that is A is nonsingular.

$$
\begin{aligned}
AX &= B && A \text{ has an inverse } A^{-1}. \\
A^{-1}(AX) &= A^{-1}B && \text{Multiply both sides by } A^{-1}. \\
(A^{-1}A)X &= A^{-1}B && \text{Apply the Associative Property on the left side.} \\
I_nX &= A^{-1}B && \text{Apply the Inverse Property: } A^{-1}A = I_n. \\
X &= A^{-1}B && \text{Apply the Identity Property: } I_nX = X.
\end{aligned}
$$

This leads to the following result:

Theorem 6.14:

A system of n linear equations containing n variables

$$\boxed{AX = B}$$

for which A is a square matrix and A^{-1} exists, always has a unique solution that is given by

$$\boxed{X = A^{-1}B}$$

Example 6.15: Solve the system of equations:

$$
\begin{aligned}
x + y + 2z &= 1 \\
2x + y &= 2 \\
x + 2y + 2z &= 3
\end{aligned}
$$

Solution: Here

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

the solution X of the system is

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -\frac{1}{2} \end{bmatrix}$$

Therefore, $S.S. = \{(0,2,\frac{-1}{2})\}$.

Application: Use Matrices in Cryptography

Cryptography is the art of writing or deciphering secret codes. We begin by giving examples of elementary codes.

Example 6.17: (Encoding a Message)

A message can be encoded by associating each letter of the alphabet with some other letter (or numbers) of the alphabet according to a prescribed pattern. For example, we might have

$$\begin{array}{cccccccccccccccccccccccccc} A & B & C & D & E & F & G & H & I & J & K & L & M & N & O & P & Q & R & S & T & U & V & W & X & Y & Z \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

Suppose we want to encode the following message:

TOP SECURITY CLEARANCE

If we decide to divide the message into pairs of letters, the message becomes

TO PS EC UR IT YC LE AR AN CE

(If there is a letter left over, arbitrarily assign Z to the last position.) Using the correspondence of letters to numbers given above, and writing each pair of letters as a column vector, we obtain

$$\begin{bmatrix} T \\ O \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix} \quad \begin{bmatrix} P \\ S \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix} \quad \begin{bmatrix} E \\ C \end{bmatrix} = \begin{bmatrix} 22 \\ 24 \end{bmatrix} \quad \begin{bmatrix} U \\ R \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix} \quad \begin{bmatrix} I \\ T \end{bmatrix} = \begin{bmatrix} 18 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} 2 \\ 24 \end{bmatrix} \quad \begin{bmatrix} L \\ E \end{bmatrix} = \begin{bmatrix} 15 \\ 22 \end{bmatrix} \quad \begin{bmatrix} A \\ R \end{bmatrix} = \begin{bmatrix} 26 \\ 9 \end{bmatrix} \quad \begin{bmatrix} A \\ N \end{bmatrix} = \begin{bmatrix} 26 \\ 13 \end{bmatrix} \quad \begin{bmatrix} C \\ E \end{bmatrix} = \begin{bmatrix} 24 \\ 22 \end{bmatrix}$$

Next, arbitrarily choose a 2×2 matrix A, which has an inverse A^{-1}.

Let's choose

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

whose its inverse is

$$A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Now transform the column vectors representing the message by multiplying each of them on the left by matrix A:

$$A\begin{bmatrix} \text{T} \\ \text{O} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 50 \\ 31 \end{bmatrix}$$

$$A\begin{bmatrix} \text{P} \\ \text{S} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 11 \\ 8 \end{bmatrix} = \begin{bmatrix} 46 \\ 27 \end{bmatrix}$$

$$A\begin{bmatrix} \text{E} \\ \text{C} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 22 \\ 24 \end{bmatrix} = \begin{bmatrix} 116 \\ 70 \end{bmatrix}$$

$$A\begin{bmatrix} \text{U} \\ \text{R} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 39 \\ 24 \end{bmatrix}$$

$$A\begin{bmatrix} \text{I} \\ \text{T} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 18 \\ 7 \end{bmatrix} = \begin{bmatrix} 57 \\ 32 \end{bmatrix}$$

$$A\begin{bmatrix} \text{Y} \\ \text{C} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 24 \end{bmatrix} = \begin{bmatrix} 76 \\ 50 \end{bmatrix}$$

$$A\begin{bmatrix} \text{L} \\ \text{E} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 15 \\ 22 \end{bmatrix} = \begin{bmatrix} 96 \\ 59 \end{bmatrix}$$

$$A\begin{bmatrix} \text{A} \\ \text{R} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 26 \\ 9 \end{bmatrix} = \begin{bmatrix} 79 \\ 44 \end{bmatrix}$$

$$A\begin{bmatrix} \text{A} \\ \text{N} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 26 \\ 13 \end{bmatrix} = \begin{bmatrix} 91 \\ 52 \end{bmatrix}$$

$$A\begin{bmatrix} \text{C} \\ \text{E} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 24 \\ 22 \end{bmatrix} = \begin{bmatrix} 114 \\ 68 \end{bmatrix}$$

The coded message is

50 31 46 27 116 70 39 24 57 32 76 50 96 59 79 44 91 52 114 68.

To decode or unscramble the above message, pair the numbers in 2×1 column vectors. Then on the left multiply each column vector by A^{-1}. For example, the first two column vectors then become

$$A^{-1}\begin{bmatrix} 50 \\ 31 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 50 \\ 31 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix} = \begin{bmatrix} \text{T} \\ \text{O} \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 46 \\ 17 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 46 \\ 27 \end{bmatrix} = \begin{bmatrix} 11 \\ 8 \end{bmatrix} = \begin{bmatrix} \text{P} \\ \text{S} \end{bmatrix} \text{ etc.}$$

Continuing in this way, the original message is obtained.