mixed strategy game: They are the games in which each player's

Strategies are mixed.

Now we want to calculate the expected payoff of a game that is not strictly determined. Let's consider a game defined by the 2x2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Let the strategy for player A, who plays rows, be denoted by $P = [P_1 \; P_2]$ and the strategy for B, who playes col, be denoted by $Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$.

The propability that A wins the amount a_{11} is $P_1 q_1$, the same thing done for the same amounts. Let $E(P,Q)$ be the expectation of player A, then

$$E(P,Q) = P_1 a_{11} q_1 + P_1 a_{12} q_2 + P_2 a_{21} q_1 + P_2 a_{22} q_2.$$

Using matrix notation, then

$$E(P,Q) = PAQ$$

Ex4: Consider a two-person zero-sum game given by $\begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix}$

Is this game strictly determined? If so, find its value.

Sol:

$$\begin{array}{cc|c} \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix} & \begin{matrix} 0 \\ -2 \end{matrix} & \text{Maxmin} \\ \begin{matrix} 6 & 3 \end{matrix} & & \\ \text{Minmax} & & \end{array}$$

notice that Maxmin ≠ Minmax

Thus this game is not strictly determined.

Ex5: Again call Ex4, if player A chooses row1 with prob. 0.5 and row2 with 0.5 and B chooses Col1 with 0.3 and Col2 with 0.7, then expected pay off is:

$$E = \underbrace{0.5}_{P_1}\underbrace{(6)}_{a_{11}}\underbrace{0.3}_{q_1} + \underbrace{0.5}_{P_1}\underbrace{(0)}_{a_{12}}\underbrace{0.7}_{q_2} + \underbrace{0.5}_{P_2}\underbrace{(-2)}_{a_{21}}\underbrace{0.3}_{q_1} + \underbrace{0.5}_{P_2}\underbrace{(3)}_{a_{22}}\underbrace{0.7}_{q_2} = 1.65$$

that makes the game favorable to A

...eral, if A is an $m \times n$ matrix game, we are led to the following definition

Expected Payoff:- The expected payoff E of a two-person zero-sum game, defined by the matrix A, which the row vector P and col. vector Q define the respective strategy prob. of player A and B is:

$$E = PAQ$$

Ex6: Find the expected payoff of the matrix game $A = \begin{bmatrix} 3 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{matrix} -1 \\ -2 \\ 0 \end{matrix}$
$\quad\quad\quad\quad 3 \quad 1$

if player A and B decide on the strategies

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad Q = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

Sol: The expected payoff E is

$$E = PAQ = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{2}{9}$$

the game is favorable to player A.

H.W.: ① For the game of Ex4, find E if player A chooses row1 in 30% and B chooses col1 in 40% of the plays.

② Find E of the game $\begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}$ of the given strategies:

ⓐ $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$; $Q = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ ⓑ $P = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}$; $Q = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

③ $\begin{bmatrix} 4 & 0 \\ -3 & 6 \end{bmatrix}$; $P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$; $Q = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

④ $\begin{bmatrix} 4 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$; $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$; $Q = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{6} \\ \frac{1}{6} \end{bmatrix}$

4-3 Optimal strategy in Two-Person Zero-Sum games with 2×2 Matrices

A player in a game may chooses his strategy randomly a ccording to the laws of prob. This is has the effect of making it impossible for the opponent to Know what the player will do.

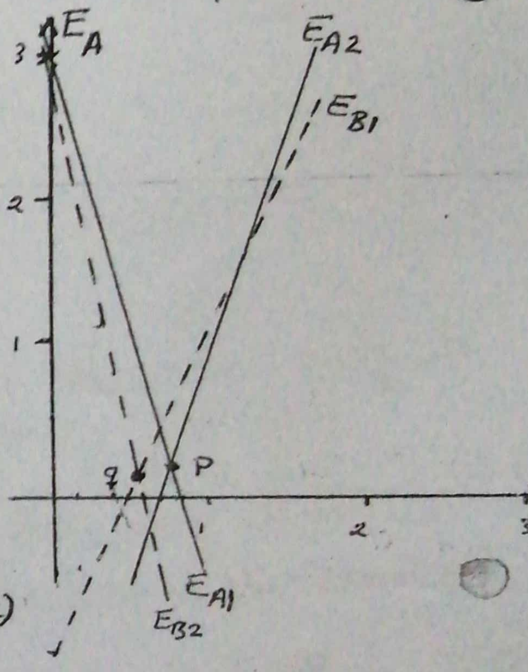
Ex7: Consider the non-Strictly determined game $A=\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}$.

Determine the optimal strategy for each player using graphical Technique.

Sol: If player A chooses raw1 with prob. P, then he must choose row2 with prob 1-P. If B chooses Col1, A then expects to earn

$1p_1+(-2)p_2$

$$E_A = P+(-2)(1-P) = 3P-2 \quad \text{-----}(1)$$



Similarly, if B chooses Col2, A expects to earn

$-1p_1+3p_2$

$$E_A = (-1)P+3(1-P) = -4P+3 \quad \text{---}(2)$$

we graph these using E_A as y-axis and p as x-axis.

A wants to maximize his earning, so he should maximize the min. expected. This occurs when that the two lines intersect. Thus, solving (1) and (2) simultaneously; we obtain:

$3P-2=-4P+3 \Rightarrow P=\frac{5}{7}$. The optimal strategy for A is $P=\left[\frac{5}{7}\ \frac{2}{7}\right]$

$P_2 = 1-P$

In same way we can get:

$$E_B = q+(-1)(1-q) = 2q-1 \quad \text{-----}(3)$$

$$E_B = (-2)q+3(1-q) = -5q+3 \quad \text{-----}(4)$$

Solving (3) and (4). we obtain: $q=\frac{4}{7}$, The optimal strategy for B is $Q=[$

The expected payoff E for this game is:

$$E = PAQ = \left[\frac{5}{7}\ \frac{2}{7}\right]\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}\begin{bmatrix}\frac{4}{7}\\ \frac{3}{7}\end{bmatrix} = \frac{1}{7}$$, Its favorable to A.

of a Game: Now consider a two-person Zero-game given by 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where $P = [P_1 \; P_2]$ and $Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, then

$$P_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \;;\quad P_2 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \quad \cdots\cdots (5)$$

and

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \;,\quad q_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \quad \cdots\cdots (6)$$

with $a_{11} + a_{22} - a_{12} - a_{21} \neq 0$, and $P_1 + P_2 = 1$, $q_1 + q_2 = 1$

The Expected payoff E of the game is:

$$E = PAQ = \frac{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \quad \cdots\cdots (7)$$

When optimal strategies are used, then E of the game is called the value V of the game. Eq. (5), (6) and (7) are called <u>Ratio technique</u>.

<u>Ex8</u>: Find the optimal strategy for each player, and determine the value of the game given by the matrix $\begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix}$ using ratio technique

<u>Sol</u>:

$$P_1 = \frac{3 - (-2)}{6 + 3 - 0 - (-2)} = \frac{5}{11} \;,\quad P_2 = \frac{6 - 0}{11} = \frac{6}{11}$$

$$q_1 = \frac{3 - 0}{11} = \frac{3}{11} \;,\quad q_2 = \frac{6 - (-2)}{11} = \frac{8}{11}$$

$V = \frac{18 - (-2) \cdot 0}{11} = 1.64$, Thus, the game is favorable to player A.

whose strategy is $P = [\frac{5}{11} \; \frac{6}{11}]$.

Ex9: (Application)

In prasidential campaign, there are two candidates, a Democrat (D) and Republican (R), and two types of issues, domestic issues (di) and foreign issues (fi). The units assigned to each candidate's strategy are given in the table. We assume that a strength (posative entry) of D equals a weakness of R and vice versa, so that the game is zero sum. What is the best strategy for each candidate. What is the value of the game.

		R: di	R: fi
D	di	4	-2
D	fi	-1	3

Sol: Notice that this game is not strictly determine.

$$P_1 = \frac{3-(-1)}{4+3-(-2)-(-1)} = \frac{4}{10} = 0.4, \quad P_2 = \frac{4-(-2)}{10} = 0.6$$

$$q_1 = \frac{3-(-2)}{10} = 0.5, \quad q_2 = \frac{4-(-1)}{10} = 0.5$$

Analysis: The best streategy for D is to spend 40% of his time on domestic issues and 60% on forign issues, while R should divide his time evenly بالتساوي between the two issues.

$$V = \frac{3\cdot 4-(-1)(-2)}{10} = \frac{10}{10} = 1$$

Thus, no matter what R does, D gains 1 unit by employing his best strategy.

Ex10: (Application - War Game)

In a naval battle, attacking bomber planes are trying to sink ships in a fleet protected by an aircraft carrier with fighter planes. The bombers can attack either high or low, with a low attack giving more accurate results. Similarly, the aircraft carrier send its fighters at high or low altitudes to search for the bombers. If the bombers avoid the fighters, credit the bombers with 8 points; if the bombers and fighters meet, credit the bombers -2 points. Also credit the bombers with 3 additional points for flying low (more accurate bombing). Find opt. strategies for the

...t, we set up the game matrix. Designate the bombers as playing ...nd the fighters as playing Cols.

$$\begin{array}{ccc} & & \text{Fighters} \\ & & \begin{array}{cc}\text{Low} & \text{High}\end{array} \\ \text{Bombers} & \begin{array}{c}\text{Low} \\ \text{High}\end{array} & \begin{bmatrix} 1 & 11 \\ 8 & -2 \end{bmatrix} \rightarrow 8+3 \end{array}$$

$$\begin{bmatrix} -2+3 & 8+3 \\ 8 & -2 \end{bmatrix}$$

$p_1 = \frac{-10}{-20} = \frac{1}{2}$, $P_2 = 1 - \frac{1}{2} = \frac{1}{2}$

$q_1 = \frac{-13}{-20} = \frac{13}{20}$, $q_2 = \frac{7}{20}$

$$V = \frac{-2-88}{-20} = \frac{-90}{-20} = 4.5$$

The game is favorable to bombers which can decide whether to fly high or low.

H.W ① Find the optimal strategies and determine the value of each 2x2 matrix by using graphical technique. Check your answer by using Ratio technique.

ⓐ. $\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$ ⓑ. $\begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix}$ ⓒ. $\begin{bmatrix} 5 & 4 \\ -3 & 7 \end{bmatrix}$

2) A spy can leave an airport through two exists, one a deserted exit and the other an exist used by the public. His opponent must guess which exit he will use. If the spy and opponent meet at the deserted exit, the spy will be killed and credit −100 points; if the two meet at the heavily used exit, the spy will be ...rested with −2 points. Assign a payoff 30 points to the spy if he avoids his opponent by using the deserted exit and of 10 points to the spy if he avoids his opponent by using the busy exit.

③ In example 9, suppose the candidates are assigned the following table. What is each candidate's best strategy? What is the value of the game and whom does it favor

		R	
		di	fi
D	di	4	0
	fi	-1	3