CH3 : Limits , Continuity and Differentiation

S3.1 : Limits and Continuity

Remark 3.1.1: If the values of a function $y = f(x)$ can be made as close as
we like to a fixed number L by taking x close to x_0 (but not equal to x_0)
we say that L is the limit of f as x approaches x_0 , and we write it as

$$\lim_{x \to x_0} f(x) = L$$

Also we can say that the limit of f as x approaches x_0 equals L .

Definition 3.1.2 :

Let f be a function defined on the set $(x_0 - p , x_0) \cup (x_0 , x_0 + p)$, with $p > 0$. Then

$$\lim_{x \to x_0} f(x) = L$$

iff for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

if $0 < |x - x_0| < \delta$ **then** $|f(x) - L| < \varepsilon$.

Theorem 1 :

1) $\lim_{x \to x_0} x = x_0$
2) $\lim_{x \to x_0} k = k$

Theorem 2 : If $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} g(x) = L_2$, then

1) $\lim_{x \to x_0} [f(x) + g(x)] = L_1 + L_2$
2) $\lim_{x \to x_0} [f(x) - g(x)] = L_1 - L_2$
3) $\lim_{x \to x_0} [f(x) \, . \, g(x)] = L_1 \, . \, L_2$
4) $\lim_{x \to x_0} [k \, . \, f(x)] = k \, . \, L_1$
5) $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$ if $L_2 \neq 0$.

Example 3.1.3 : Find each of the following :

1. $\lim_{x \to -2} 7$

2. $\lim_{x \to 1} x\,(3 - x)$

3. $\lim_{x \to 3} (x^2 + 2x - 1)$

4. $\lim_{x \to 2} \dfrac{x - 2}{x^2 - 5x + 6}$

5. $\lim_{x \to 0} \dfrac{x^2 - 5x}{x}$

Solution :

1. $\lim_{x \to -2} 7 = 7$

2. $\lim_{x \to 1} x\,(3 - x) = 1(3 - 1) = 2$

3. $\lim_{x \to 3} (x^2 + 2x - 1) = (3)^2 + 2(3) - 1 = 9 + 6 - 1 = 14$

4. $\lim_{x \to 2} \dfrac{x - 2}{x^2 - 5x + 6} = \lim_{x \to 2} \dfrac{x - 2}{(x - 3)(x - 2)} = \lim_{x \to 2} \dfrac{1}{(x - 3)} = \dfrac{1}{2 - 3} = -1$

5. $\lim_{x \to 0} \dfrac{x^2 - 5x}{x} = \lim_{x \to 0} \dfrac{x(x - 5)}{x} = \lim_{x \to 0} (x - 5) = 0 - 5 = -5$

Theorem 3 :

1) $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

2) $\lim_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

Example 3.1.4 : Find each of the following :

1. $\lim_{x \to 0} \dfrac{\sin 4x}{\sin 5x}$

2. $\lim_{x \to 0} \dfrac{3x}{\sin 2x}$

3. $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$

Solution :

1. $\lim\limits_{x \to 0} \dfrac{\sin 4x}{\sin 5x} = \lim\limits_{x \to 0} \dfrac{4x \,.\, \dfrac{\sin 4x}{4x}}{5x \,.\, \dfrac{\sin 5x}{5x}} = \dfrac{4}{5}$

2. $\lim\limits_{x \to 0} \dfrac{3x}{\sin 2x} = \lim\limits_{x \to 0} \dfrac{3x}{2x \,.\, \dfrac{\sin 2x}{2x}} = \dfrac{3}{2}$

3. $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = \lim\limits_{x \to 0} \dfrac{\dfrac{\sin x}{\cos x}}{\dfrac{x}{1}} = \lim\limits_{x \to 0} \left(\dfrac{\sin x}{\cos x} \,.\, \dfrac{1}{x} \right) = \lim\limits_{x \to 0} \left(\dfrac{\sin x}{x} \,.\, \dfrac{1}{\cos x} \right)$

$= 1 \times 1 \quad = 1$

Exercise 3.1.5 : Find each of the following :

1. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x + \sin x}$

2. $\lim\limits_{x \to \infty} \left(1 + \cos \frac{1}{x}\right)$

3. $\lim\limits_{x \to 0} \dfrac{\sin 2x}{2x^2 + x}$

4. $\lim\limits_{y \to 0} \dfrac{\tan 2y}{3y}$

5. $\lim\limits_{y \to \infty} \dfrac{y^4}{y^4 - 7y^3 + 3y^2 + 9}$

Definition 3.1.6 : A function $f(x)$ is said to be continuous at x_0 if

1) f is defined at x_0 (i.e. $f(x_0) = \mathrm{L}$ where $\mathrm{L} \in \mathrm{R}$).

2) $\lim\limits_{x \to x_0} f(x)$ exists

3) $\lim\limits_{x \to x_0} f(x) = f(x_0) = \mathrm{L}$

Example 3.1.7 : Let $f(x) = \begin{cases} x^2 & x \leq 1 \\ 3 - 2x & x > 1 \end{cases}$

Is f continuous at $x = 1$.

Solution :

1) $f(1) = 1^2 = 1$

2) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = 1^2 = 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (3 - 2x) = 3 - 2(1) = 1$

since $\lim\limits_{x \to 1^-} f(x) = 1 = \lim\limits_{x \to 1^+} f(x)$

Therefore $\lim\limits_{x \to 1} f(x)$ exists and $\lim\limits_{x \to 1} f(x) = 1$

3) $\lim\limits_{x \to 1} f(x) = 1 = f(1)$

Therefore f is continuous at $x = 1$

Example 3.1.8 : Let $f(x) = \begin{cases} 2x + 1 & \text{if } x < -2 \\ x^2 - 2 & \text{if } x \geq -2 \end{cases}$

Is f continuous at $x = -2$.

Solution :

1) $f(-2) = (-2)^2 - 2 = 4 - 2 = 2$

2) $\lim\limits_{x \to (-2)^-} f(x) = \lim\limits_{x \to (-2)^-} (2x + 1) = 2(-2) + 1 = -4 + 1 = -3$

$\lim\limits_{x \to (-2)^+} f(x) = \lim\limits_{x \to (-2)^+} (x^2 - 2) = (-2)^2 - 2 = 4 - 2 = 2$

since $\lim\limits_{x \to (-2)^-} f(x) \neq \lim\limits_{x \to (-2)^+} f(x)$

Therefore $\lim\limits_{x \to (-2)} f(x)$ does not exists

Thus f is not continuous at $x = -2$.

Exercise 3.1.9 :

Let $f(x) = \begin{cases} \frac{x^2-2x-8}{x+2} & if \ x \neq -2 \\ -3 & if \ x = -2 \end{cases}$

Is f continuous at $x = -2$.

S3.2 : Differentiation

Definition of Derivative , Rules of Differentiation

Definition 3.2.1:

Let $y = f(x)$ be a function and let the variable x receive a certain increment Δx. Then the function y will receive a certain increment Δy. Thus for the value of x we have $y = f(x)$ and for the value of $x + \Delta x$, we have $y + \Delta y = f(x + \Delta x)$.

Thus the increment Δy is given by :

$$\Delta y = f(x + \Delta x) - f(x)$$

Remark 3.2.2 : Δ is an abbreviation of difference (in x, y) and is not a factor .

Forming the ratio of the increment of the function y to the increment of the variable x , we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is called the average rate of change of the function $y = f(x)$ with respect to the variable x . $\frac{\Delta y}{\Delta x}$ is also called the difference quotient of the function $y = f(x)$. If the limit of this ratio as Δx approaches zero exists, that is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exist, then the function is called differentiable and the limit $(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x})$ is called the first derivative of the function $y = f(x)$ with respect to the variable x, which is denoted by $f'(x)$, y', $\frac{dy}{dx}$, $\frac{d}{dx}y$, $\frac{d}{dx}f(x)$.

Differentiation Rules:

Let $f(x)$ and $g(x)$ be two differentiable functions (in the interval under consideration) , then

RULE 1 Constant Multiple Rule

If $f(x)$ is a differentiable function of x, and c is a constant , then

$$\frac{d}{dx}(\,c\,f(x)\,) = c\,\frac{d}{dx}\,f(x)\,.$$

RULE 2 Derivative of the Sum

If $f(x)$ and $g(x)$ are differentiable functions of x, then their sum $f(x) + g(x)$ is differentiable , and

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}\,f(x) + \frac{d}{dx}\,g(x)$$

RULE 3 Derivative of the Difference

If $f(x)$ and $g(x)$ are differentiable functions of x, then their difference $f(x) - g(x)$ is differentiable , and

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}\,f(x) - \frac{d}{dx}\,g(x)$$

RULE 4 Derivative of the Product

If $f(x)$ and $g(x)$ are differentiable functions of x, then their product $f(x).\,g(x)$ is differentiable , and

$$\frac{d}{dx}(f(x).g(x)) = f(x).\frac{d}{dx}\,g(x) + g(x).\frac{d}{dx}\,f(x)$$

RULE 5 Derivative of the Quotient

If $f(x)$ and $g(x)$ are differentiable functions of x and $g(x) \neq 0$, then the quotient $\dfrac{f(x)}{g(x)}$ is differentiable , and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).\dfrac{d}{dx}f(x) - f(x).\dfrac{d}{dx}g(x)}{g(x)^2}$$

Derivatives of Some Special Functions and the Chain Rule:

1) Derivatives of Some Algebraic Functions:

1) Derivative of a Constant Function

If $f(x) = c$, then $\dfrac{d}{dx}f(x) = \dfrac{d}{dx}c = 0$

Example 3.2.3 : If $f(x) = 12$, then $\dfrac{d}{dx}f(x) = \dfrac{d}{dx}(12) = 0$.

2) Derivatives of a Power Functions

$$\frac{d}{dx}x^n = n\,x^{n-1} \quad , \quad n \in Q$$

provided that $x \neq 0$ when n is negative .

Example 3.2.4 : Find f' for each of the following functions :

(i) $f(x) = x$, (ii) $f(x) = x^2$, (iii) $f(x) = x^{-3}$, (iv) $f(x) = x^{0.3}$

Solution:

(i) $f'(x) = x^{1-1} = x^0 = 1$

(ii) $f'(x) = 2x^{2-1} = 2x$

(iii) $f'(x) = -3x^{-3-1} = -3x^{-4}$

(iv) $f'(x) = 0.3x^{0.3-1} = 0.3x^{-0.7}$

Example 3.2.5 : Find f' for each of the following functions :

$(i)\ f(x)=\frac{1}{2}x$, $(ii)\ f(x)=9x^2$, $(iii)\ f(x)=4x^{-3}$, $(iv)\ f(x)=x^{2.5}$,

Solution:

$(i)\ f'(x)=\frac{1}{2}(1)=\frac{1}{2}$

$(ii)\ f'(x)=9(2x^{2-1}) = 18x$

$(iii)\ f'(x)=4((-3)x^{-3-1}) = -12x^{-4}$

$(iv)\ f'(x)=2.5\,x^{2.5-1}=2.5\,x^{1.5}$

Example 3.2.6 : Find f' for each of the following functions :

$(i)\ f(x)= x^2+5x^{-3}$, $(ii)\ f(x)=x^4-\frac{3}{5}x^2+7x-14$

Solution:

$(i)\ f'(x) = 2x - 15x^{-4}$

$(ii)\ f'(x)=4x^3-\frac{3(2)}{5}x + 7 - 0 = 4x^3-\frac{6}{5}x+7$

Example 3.2.7 : Find f' for the function $f(x)=2x(3x^5+\frac{3}{x})$

Solution:

$$f'(x)= 2x(15x^4-\frac{3}{x^2})+(3x^5+\frac{3}{x}).2$$

$$= 30x^5-\frac{6}{x}+6x^5+\frac{6}{x} = 36x^5$$

Example 3.2.8 : Find f' for the function $f(x)=\frac{2x-1}{3x+1}$

Solution:

$$f'(x) = \frac{(3x+1).2-(2x-1).3}{(3x+1)^2}$$

$$= \frac{6x+2-6x+3}{(3x+1)^2} = \frac{5}{(3x+1)^2}$$

The derivative of the cosine function is the negative of the sine function :

$$\frac{d}{dx}(\cos x) = -\sin x$$

Example 3.2.10 : Find $f'(x)$ for the function $f(x) = 3x^2 + 2\cos x$

Solution: $f'(x) = 6x - 2\sin x$

Example 3.2.11 : Find y' for each of the following functions :

$(i)\ y = \sin x - \cos x$ $\quad (ii)\ y = 2\sin x\ \cos x$ $\quad (iii)\ y = \dfrac{3\sin x}{\cos x + 1}$

Solution:

$(i)\ y' = \cos x + \sin x$

$(ii)\ y' = 2\sin x\,.\,(-\sin x) + \cos x\,.\,(2\cos x) = -2\sin^2 x + 2\cos^2 x$

$$(iii)\ y' = \frac{(\cos x + 1)\,.\,(3\cos x) - (3\sin x)\,.\,(-\sin x)}{(\cos x + 1)^2}$$

$$= \frac{3\cos^2 x + 3\cos x + 3\sin^2 x}{(\cos x + 1)^2}$$

The derivative of other trigonometric functions :

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 3.2.12 : Find y' for each of the following functions :

$(i)\ y = \tan x + \sec x$ $\quad (ii)\ y = 5\cot x\ \csc x$

Solution:

$(i)\ y' = \sec^2 x + \sec x \tan x$

$(ii)\ y' = 5\cot x \,.\, (-\csc x \cot x) + \csc x \,.\, (-5\csc^2 x)$

$= -5\csc x \cot^2 x - 5\csc^3 x$

Derivative of Logarithmic Function:

> The derivative of the natural logarithmic function is:
>
> $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Example 3.2.13 : Find y' for each of the following functions :

$(i)\ y = 4x^3 \ln x$ $\qquad (ii)\ y = \dfrac{2\ln x}{9x+1}$

Solution:

(i) $y' = 4x^3 (\frac{1}{x}) + \ln x (12x^2) = 4x^2 + 12x^2 \ln x$

$$(ii)\ y' = \frac{(9x+1)(\frac{2}{x}) - (2\ln x)(9)}{(9x+1)^2} = \frac{18 + \frac{2}{x} - 18\ln x}{(9x+1)^2}$$

Derivative of Exponential Function :

> The derivative of the exponential functions are:
>
> $$\frac{d}{dx}a^x = a^x \ln a \quad \text{and} \quad \frac{d}{dx}e^x = e^x$$

Example 3.2.14 : Find f' for the function $f(x) = 5x^7 e^x + 4e^x$.

Solution: $f'(x) = 5x^7 e^x + e^x (35x^6) + 4e^x = 5x^7 e^x + 35x^6 e^x + 4e^x$

Implicit Differentiation (Derivative of Composite Functions) :

Chain Rule :

Let $y = f(u)$, $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Example 3.2.15 : Let $y = 6u^3 + 5u$, $u = \ln x$, find $\dfrac{dy}{dx}$.

Solution:

$$\frac{dy}{du} = 18u^2 + 5 \quad , \quad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (18u^2 + 5)(\frac{1}{x}) = \left(18\,(\ln x)^2 + 5\right)(\frac{1}{x})$$

$$= \frac{18}{x}(\ln x)^2 + \frac{5}{x}$$

Example 3.2.16 : Find $\dfrac{dy}{dx}$ for each of the following functions :

(i) $y = (x + 4x^3)^6$, (ii) $y = \ln(x^2 + 3)$, (iii) $y = \tan^3 x$.

Solution:

(i) let $u = x + 4x^3$, then $y = u^6$.

Thus $\dfrac{dy}{du} = 6u^5$ and $\dfrac{du}{dx} = 1 + 12x^2$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6u^5(1 + 12x^2) = 6(x + 4x^3)^5(1 + 12x^2) \; .$$

(ii) let $u = x^2 + 3$, then $y = \ln u$.

Thus $\dfrac{dy}{du} = \dfrac{1}{u}$ and $\dfrac{du}{dx} = 2x$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u}(2x) = \frac{2x}{x^2 + 3}$$

(iii) let $u = \tan x$, then $y = u^3$.

Thus $\dfrac{dy}{du} = 3u^2$ and $\dfrac{du}{dx} = \sec^2 x$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2\,(\sec^2 x) = 3\tan^2 x \, \sec^2 x$$

In examples (3.2.15 and 3.2.16) we use the Chain rule to get the derivative of a composite function using substitutions, but also we can get the same results directly without substitutions, considering the following rules:

$$\frac{d}{dx}(f(x))^n = n\,(f(x))^{n-1} \,.\, f'(x),$$

$$\frac{d}{dx}(\ln f(x)) = \frac{1}{f(x)} \,.\, f'(x),$$

$$\frac{d}{dx}(e^{f(x)}) = e^{f(x)} \,.\, f'(x),$$

$$\frac{d}{dx}(\sin f(x)) = (\cos f(x)) \,.\, f'(x),$$

$$\frac{d}{dx}(\cos f(x)) = (-\sin f(x)) \,.\, f'(x),$$

$$\frac{d}{dx}(\tan f(x)) = (\sec^2 f(x)) \,.\, f'(x),$$

$$\frac{d}{dx}(\sec f(x)) = (\sec f(x) \,.\, \tan f(x)) \,.\, f'(x),$$

$$\frac{d}{dx}(\csc f(x)) = (-\csc f(x) \,.\, \cot f(x)) \,.\, f'(x),$$

$$\frac{d}{dx}(\cot f(x)) = (-\csc^2 f(x)) \,.\, f'(x).$$

<u>Example 3.2.17 :</u> Find $\frac{dy}{dx}$ for each of the following functions :

$(i)\ y=\sqrt{x^5+4x}$, $(ii)\ y=\ln(x^2+3x)$, $(iii)\ y=e^{3x}$.

<u>Solution:</u>

$(i)\ y=\sqrt{x^5+4x} = (x^5+4x)^{\frac{1}{2}}$

$$\therefore\ \frac{dy}{dx} = \frac{1}{2}(x^5+4x)^{-\frac{1}{2}} \,.\, (5x^4+4) = \frac{5x^4+4}{2\sqrt{x^5+4x}} \ .$$

(ii) $y = \ln(x^2 + 3x)$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2+3x} \cdot (2x+3) = \frac{2x+3}{x^2+3x} .$$

(iii) $y = e^{3x}$

$$\therefore \frac{dy}{dx} = e^{3x} \cdot 3 = 3e^{3x} .$$

Second Order Derivative and Derivatives of Higher Order:

When we differentiate a function $y = f(x)$ we get a new function y' (or $\frac{dy}{dx}$ or $f'(x)$ or $\frac{d}{dx}f$) which is the derivative of $y = f(x)$ (or the first derivative of $y = f(x)$) . Now if this derivative $y' = f'(x)$ is also a differentiable function , we can define the second derivative of $y = f(x)$ (or the second order derivative of $y = f(x)$) by differentiating y' (or $\frac{dy}{dx}$ or $f'(x)$ or $\frac{d}{dx}f$) , which is denoted by y'' (or $\frac{d^2y}{dx^2}$ or $f''(x)$ or $\frac{d^2}{dx^2}f$).

Now if the second derivative $y'' = f''(x)$ is also a differentiable function , we can define the third derivative of $y = f(x)$ (or the third order derivative of $y = f(x)$) by differentiating y'' (or $\frac{d^2y}{dx^2}$ or $f''(x)$ or $\frac{d^2}{dx^2}f$) , which is denoted by y''' (or $\frac{d^3y}{dx^3}$ or $f'''(x)$ or $\frac{d^3}{dx^3}f$) . So long as we have differentiability , we can continue in this manner forming the fourth derivative of $y = f(x)$, which is denoted by $y^{(4)}$ (or $\frac{d^4y}{dx^4}$ or $f^{(4)}(x)$ or $\frac{d^4}{dx^4}f$) , and more generally the nth derivative of $y = f(x)$ is denoted by $y^{(n)}$ (or $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$ or $\frac{d^n}{dx^n}f$).

<u>Example 3.2.18 :</u> Find y'' for each of the following functions :

(i) $y = 4x^5 - 7x^3 + 3x$, (ii) $y = x^3 e^{4x}$, (iii) $y = 2\sin x + 9\cos x$

Solution:

$(i)\ y' = 20x^4 - 21x^2 + 3 \quad , \quad y'' = 80x^3 - 42x$

$(ii)\ y' = x^3 (4e^{4x}) + e^{4x}(3x^2) = 4x^3 e^{4x} + 3x^2 e^{4x}$

$$y'' = 4x^3(4e^{4x}) + e^{4x}(12x^2) + 3x^2(4e^{4x}) + e^{4x}(6x)$$

$$= 16x^3 e^{4x} + 12x^2 e^{4x} + 12x^2 e^{4x} + 6x\, e^{4x}$$

$$= 16x^3 e^{4x} + 24x^2 e^{4x} + 6x\, e^{4x}$$

$(iii)\ y' = 2\cos x - 9\sin x$

$$y'' = -2\sin x - 9\cos x$$

Example 3.2.19 : Find y' , y'' , y''' and $y^{(4)}$ for each of the following functions :

$(i)\ y = x^6 + x^4 - 3x^3 \quad , \quad (ii)\ y = e^{2x} \quad , \quad (iii)\ y = \sin x \quad , \quad (iv)\ y = \cos x$

Solution:

$(\mathrm{i})\ y' = 6x^5 + 4x^3 - 9x^2 \quad , \quad y'' = 30x^4 + 12x^2 - 18x$

$, \quad y''' = 120x^3 + 24x - 18 \quad , \quad y^{(4)} = 360x^2 + 24$.

$(ii)\ y' = 2e^{2x} \quad , \quad y'' = 4e^{2x} \quad , \quad y''' = 8e^{2x} \quad , \quad y^{(4)} = 16e^{2x}$

$(iii)\ y' = \cos x \quad , \quad y'' = -\sin x \quad , \quad y''' = -\cos x \quad , \quad y^{(4)} = \sin x$

$(iv)\ y' = -\sin x \quad , \quad y'' = -\cos x \quad , \quad y''' = \sin x \quad , \quad y^{(4)} = \cos x$

S3.3 : L'Hopital Rule

Suppose that $f(x_0) = g(x_0) = 0$, and both $f'(x_0)$ and $g'(x_0)$ exist . Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)} \quad \text{if} \quad g'(x_0) \neq 0 .$$

Example 3.3.1 : Find each of the following limits by using L'Hopital rule :

1. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{4x^3 - x - 3}$

2. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x + x^2}$

3. $\lim\limits_{x \to 0} \dfrac{3x - \sin x}{x}$

4. $\lim\limits_{x \to 0} \dfrac{\sqrt{4+x} - 2}{x}$

Solution:

1. $$\lim_{x \to 1} \frac{x^3 - 1}{4x^3 - x - 3} = \lim_{x \to 1} \frac{3x^2}{12x^2 - 1} = \frac{3}{12 - 1} = \frac{3}{11}$$

2. $$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \to 0} \frac{0 + \sin x}{1 + 2x} = \frac{\sin 0}{1 + 0} = \frac{0}{1} = 0$$

3. $$\lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = \frac{3 - \cos 0}{1} = \frac{3 - 1}{1} = 2$$

4. $$\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{4+x}}}{1} = \frac{\frac{1}{4}}{1} = \frac{1}{4}$$

Example 3.3.2 : Find each of the following limits by using L'Hopital rule :

1. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x \sin x}$

2. $\lim\limits_{x \to 0} \dfrac{x^4 - 5x^2}{x^2 + x - \sin x}$

Solution:

1. $$\lim_{x \to 0} \frac{x - \sin x}{x \sin x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{x \cos x + \sin x} \qquad \text{still } \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\sin x}{-x \sin x + \cos x + \cos x} = \frac{0}{2} = 0$$

2. $$\lim_{x \to 0} \frac{x^4 - 5x^2}{x^2 + x - \sin x} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{4x^3 - 10x}{2x + 1 - \cos x} \qquad \text{still } \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{12x^2 - 10}{2 + \sin x} = \frac{-10}{2} = -5 \quad .$$

Exercises :

In exercises 1 – 6 , find y' and y'' (the first and second derivatives with respect to x).

1) $y = x^3 + 6x - 5$

2) $y = 3x^4 - \dfrac{6}{x^2}$

3) $y = 7x^2 - 3\sin x$

4) $y = 5\sin x \cos x$

5) $y = 3\tan x + 4\sec x$

6) $y = 2\sin x - 5\cos x$

In exercises 7 – 9 , find the first and second derivatives of the given function with respect to the given variable.

7) $w = 2u^4 - 3u + 1$

8) $y = 6t^4 - \dfrac{4}{t}$

9) $v = t^2 - 8\sin t$

In exercises 10 – 12 , find y' by applying the Product Rule

10) $y=(4+x)(x^3-2)$

11) $y=(x+2)(x^3+x-4)$

12) $y=(4+x)(x^2-\frac{3}{x})$

In exercises 13 – 17 , find y' .

13) $y=\tan x-3\sin x$

14) $y=5\sin 3x^2+\sqrt{x}$

15) $y=3\sin x-e^x$

16) $y=\dfrac{2\sin x}{3x}$

17) $y=\dfrac{2\tan x-3x}{3x+4}$

In exercises 18 – 21 , find y', y'', y''', and $y^{(4)}$.

18) $y=x^5+6x^4-25x$

19) $y=3\sin x$

20) $y=\cos 2x$

21) $y=e^{3x}+\ln x$

In exercises 22 – 24 , find the limit by using L'Hopital rule .

22) $\lim\limits_{x\to 1}\dfrac{x-1}{3x^3-x^2-2}$

23) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x}$

24) $\lim\limits_{x\to 0}\dfrac{e^x-1}{\sin x}$

S3.4 : Applications of Derivatives:

Slope and Tangent Line and Normal Line :

The slope of the curve $y = f(x)$ at any point $P(x, y)$ is $y' = f'(x)$.

The tangent line to the curve $y = f(x)$ at any point $P_0(x_0, f(x_0))$ is the line whose equation $\frac{y - f(x_0)}{x - x_0} = f'(x_0)$ which pass through the point P_0 on the curve $y = f(x)$.

The normal line to the curve $y = f(x)$ at any point $P_0(x_0, f(x_0))$ is the line whose equation $\frac{y - f(x_0)}{x - x_0} = -\frac{1}{f'(x_0)}$ which pass through the point P_0 on the curve $y = f(x)$.

Example 3.4.1 : Find the slope of the curve of the function $y = f(x) = x^3 - 2x^2 + 4$ at the point $(1,3)$. Then find the equation of each of the tangent line and the normal line to the curve at the point $(1,3)$.

Solution :

The slope at any point $= f'(x) = 3x^2 - 4x$

$\therefore$ The slope at the point $(1,3) = f'(1) = 3 - 4 = -1$.

$$\frac{y - f(1)}{x - 1} = f'(1) \Rightarrow \frac{y - 3}{x - 1} = -1$$

$$y - 3 = -x + 1 \quad \Rightarrow \quad y + x - 4 = 0$$

Thus the equation of the tangent line at the point $(1,3)$ is $y + x - 4 = 0$.

$$\frac{y - f(1)}{x - 1} = -\frac{1}{f'(1)} \Rightarrow \frac{y - 3}{x - 1} = 1$$

$$y - 3 = x - 1 \quad \Rightarrow \quad y - x - 2 = 0$$

Thus the equation of the normal line at the point $(1,3)$ is $y - x - 2 = 0$.

Example 3.4.2 : Find the slope of the curve of the function $y = g(x) = x^2$ at the point $(3,9)$. Then find the equation of each of the tangent line and the normal line to the curve at the point $(3,9)$.

Solution : $g'(x) = 2x$

The slope of the curve at the point $(3,9)$ is $g'(3) = 2(3) = 6$.

$$\frac{y - g(3)}{x - 3} = g'(3) \quad \Rightarrow \quad \frac{y - 9}{x - 3} = 6 \quad \Rightarrow$$

$$y - 9 = 6x - 18 \quad \Rightarrow \quad y - 6x + 9 = 0$$

Thus the equation of the tangent line at the point $(3,9)$ is $y - 6x + 9 = 0$.

$$\frac{y - g(3)}{x - 3} = -\frac{1}{g'(3)} \quad \Rightarrow \quad \frac{y - 9}{x - 3} = -\frac{1}{6}$$

$$\Rightarrow \quad 6y - 54 = -x + 3 \quad \Rightarrow \quad 6y + x - 57 = 0$$

Thus the equation of the normal line at the point $(3,9)$ is $6y + x - 57 = 0$.

Exercise 3.4.3 : Find the slope of the curve of the function $y = h(x) = 3x^2 - 1$ at the point $(-1,2)$. Then find the equation of each of the tangent line and the normal line to the curve at the point $(-1,2)$.

Exercise 3.4.4 : Find the slope of the curve of the function $y = f(x) = x^3 - 4$ at the point $(2,4)$. Then find the equation of each of the tangent line and the normal line to the curve at the point $(2,4)$.