وزارة التعليم العالي والبحث العلمي
الجامعة المستنصرية
كلية التربية/ قسم الحاسبات
الدراسة الصباحية والمسائية

رياضيات الحاسبات

المرحلة الاولى

كلية التربية ابن الهيثم　　قسم علوم الحاسبات　　المرحلة الدراسية(الصف):الاول

المفردات	الساعات والوحدات			الاسم بالانكليزي	اسم المقرر
	و	ع	ن		
Definition of programming Language rules of writing prog am variables and constants // Input and output statements.arthmetic assign state men//mathematical functions and using math –h//Counters increment and decrement// Comprisal operation and logical operations and bitwise operation//Switch statement//If and nested if statement//While loop statement//For loop statement//Applications//nested loops//Break and continue//Onedimensional array//Applications on one dimensional array//Two dimensional array//Applications on two dimensional array// Multi dim sional array//Functionsfirst type//Applications//functions second type//Applications//Strrings//Structures//Arrayof structures//Passing structures to function	٤	٢	٣	Structured programming	البرمجة المهيكلة
Funcuon *Functions kinds there graphs*Limits *Continuity//Derivative(differentiation) *Fintes*Derivative by definition*Derivative by rules*Derivatives of higher order*Chain rules*Applications of derivatives// Sequen es&series*Sequences&series*Taylor&Maclurian series//Integration*Integral*Definite integral//Introductionto differentional Eq//Special functions (Gamma –Beta –Error)//*Fourier series*Fourier transformations(Ft)*Discrete ft-parserals relation properties of (Ft)*Fast Fourier transforms (FFT)//Transcecendental Function*Nature logarithm *Exponential function(e)*Exponenual function (a)*Normal logarithm *Trigonometric function&there graghs*Inverse trigonometric function*Hyperbolic function*Inverse Hyperbolic function// Polar coordinates	٣		٣	Mathematics	رياضيات
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CH2: Functions

S2.1 : Functions and Their Graphs

Definition : A function f (or a mapping f) from a set A to a set B is a rule that assigns to each element a of A exactly one element b of B. The set A is called the domain of f and the set B is called the codomain of f. If f assigns b to a, then b is called the image of a under f. The subset of B comprised of all the images of elements of A under f (which is denoted by $f(A)$) is called the image of A under f (or the range of f) .

We use $f : A \to B$ to mean that f is a function from A to B. We will write $f(a) = b$ to indicate that b is the image of a under f .

Example 2.1.1:

Let $A = \{ 2, 4, 5 \}$, $B = \{ 1, 2, 3, 6 \}$, and $f : A \to B$ be the function defined by $f(2) = 1$, $f(4) = 3$, $f(5) = 6$. Then the domain of f is $A = \{ 2, 4, 5 \}$, the codomain of f is $B = \{ 1, 2, 3, 6 \}$, and the range of $f = \{ 1, 3, 6 \}$.

Counter example :

Let $C = \{ 1, 2, 3, 4 \}$ and $D = \{ 2, 3, 4, 5 \}$, and let h be the rule defined by $h(1) = 2$, $h(1) = 4$, $h(2) = 3$, $h(3) = 5$, $h(4) = 4$, then h is not a function from C to D since there are two different elements (2 and 4) belong to D are assigned to the same element 1 of C .

Example 2.1.2: Find the domain and the range of the function f defined by $f(x) = \sqrt{x+10}$.

Solution : For $y = f(x) = \sqrt{x+10}$ to be real , $x + 10$ must be greater than or equal to 0 . That is , $x+10 \geq 0$ which means that $x \geq -10$. Thus the domain is $\{ x : x \geq -10 \}$ and the range is $\{ y : y \geq 0 \}$.

Exercises:

1) Let $A = \{ 2, 4, 5, 7 \}$, $B = \{ 1, 2, 3, 6, 9 \}$, and $f : A \to B$ be the function defined by $f(2) = 9$, $f(4) = 3$, $f(5) = 6$, $f(7) = 2$. Find the domain of f , the codomain of f , and the range of f .

2) Let f *be a function* defined by $f(x) = \dfrac{1}{x+2}$. Find the domain and the range of the function f .

3) Find the domain and the range of the function f defined by $f(x) = \sqrt{2x-9}$.

Definition: The graph of a function f is the line passing through all the points $(x, f(x))$ on the xy-plane .

Definition: The y-coordinate of the point where a graph of a function intersect the y-axis is called the y-intercept of the function

Definition: The x-coordinate of a point where a graph of a function intersects the x-axis is called an x-intercept of the function .

Remarks :

1) The graph of any function f has at most one y-intercept . The graph of the function f has exactly one y-intercept if 0 is in the domain of the function f and the y-intercept is $f(0)$.

2) The graph of any function f has no x-intercept if there is no x in the domain of the function f such that $f(x)=0$.
The graph of a function f has one or more than one x-intercepts if $f(x)=0$ for some x in the domain of f , and the number of x-intercepts is the number of the distinct solutions of the equation $f(x)=0$.

Properties of Functions :

1) A function $y = f(x)$ is called an even function of x if $f(-x) = f(x)$, $\forall\ x$.

2) A function $y = f(x)$ is called an odd function of x if $f(-x) = -f(x)$, $\forall\ x$.

S2.2 : Linear Functions and their Graphs

Definition: A function $f : R \to R$ is called a linear function if f is defined by $f(x) = ax + b$, $a \neq 0$
where a and b are real numbers .

Example 2.2.1: The function $f: R \to R$ defined by $f(x) = 3x + 12$ is a linear function .

Example 2.2.2: The function $g: R \to R$ defined by $g(x) = x - 0.2$ is a linear function .

Example 2.2.3: The function $h: R \to R$ defined by $h(x) = -\frac{3}{2}x + 1$ is a linear function .

Example 2.2.4: Let $f: R \to R$ be the linear function defined by $f(x) = 4x + 10$. Find the x-intercept and the y-intercept of f .

Solution: $f(x) = 0 \Rightarrow 4x + 10 = 0$

$$\Rightarrow 4x = -10$$

$$\Rightarrow x = -\frac{10}{4} = -2.5$$

Therefore the x-intercept is -2.5

$f(0) = 10 \Rightarrow$ the y-intercept is 10 .

Example 2.2.5: Let $g: R \to R$ be the linear function defined by $g(x) = \frac{1}{5}x - 6$. Find the x-intercept and the y-intercept of g .

Solution: $g(x) = 0 \Rightarrow \frac{1}{5}x - 6 = 0$

$$\Rightarrow \frac{1}{5}x = 6 \Rightarrow x = 30$$

Therefore the x-intercept is 30

$g(0) = -6 \Rightarrow$ the y-intercept is -6 .

Graph of a linear function :

The graph of a linear function f is the straight line passing through the two points $(a, 0)$ and $(0, b)$ where a is the x-intercept of the function f and b is the y-intercept of the function f .

Remark : The graph of any linear function f has exactly one x-intercept and has exactly one y-intercept .

Example 2.2.6: Let $f: R \to R$ be the linear function defined by $f(x) = -2x+7$. Find the x - intercept and the y - intercept of f , then graph the function f .

Solution: $f(x)=0 \Rightarrow -2x+7 = 0$

$$\Rightarrow -2x = -7$$

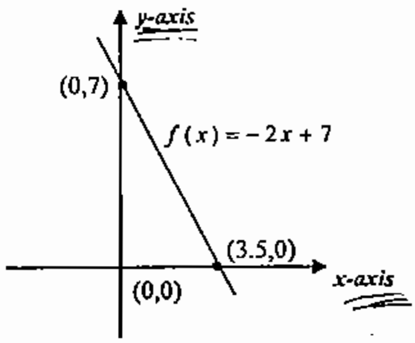
$$\Rightarrow x = \frac{-7}{-2} = 3.5$$

Therefore the x - intercept is 3.5 .

$f(0) = 7 \Rightarrow$ the y - intercept is 7 .

Thus the graph of the function f is the straight line passing through the two points $(3.5, 0)$ and $(0, 7)$.

Thus the graph of the function f is the following graph



Example 2.2.7: Let $g: R \to R$ be the linear function defined by $g(x) = 4x+12$. Find the x - intercept and the y - intercept of g , then graph the function g .

Solution: $g(x)=0 \Rightarrow 4x+12 = 0$

$$\Rightarrow 4x = -12$$

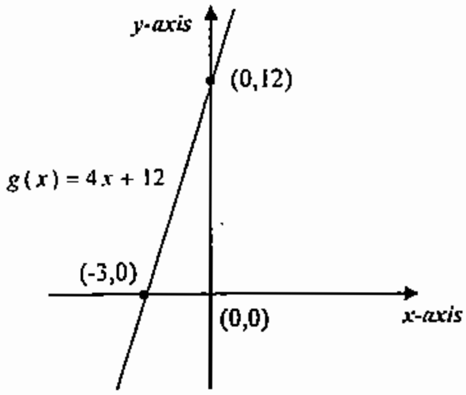
$$\Rightarrow x = \frac{-12}{4} = -3$$

Therefore the x - intercept is -3

$g(0) = 12 \Rightarrow$ the y - intercept is 12 .

Thus the graph of the function g is the straight line passing through the two points $(-3, 0)$ and $(0, 12)$.

Thus the graph of the function g is the following graph



<u>Exercises:</u>

1) Let $f : R \to R$ be the linear function defined by $f(x) = 3x - 10$.
   Find the x - intercept and the y - intercept of f .

2) Let $g : R \to R$ be the linear function defined by $g(x) = 0.3x + 0.7$.
   Find the x - intercept and the y - intercept of g .

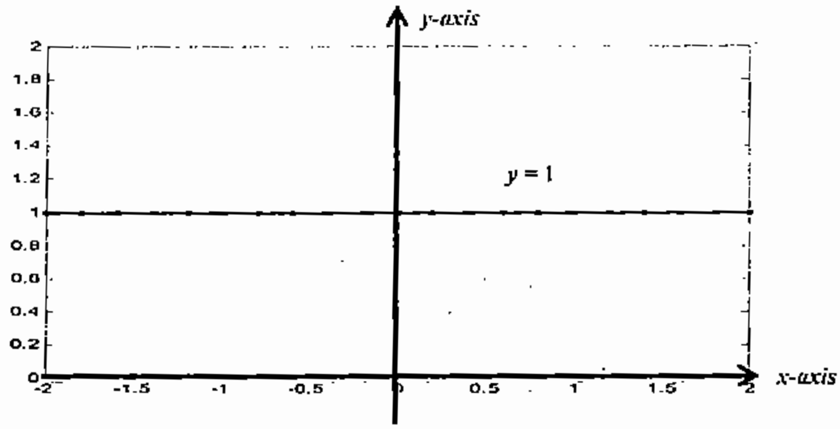
3) Let $f : R \to R$ be the linear function defined by $f(x) = -4x + 8$.
   Find the x - intercept and the y - intercept of f , then graph the function f .

4) Let $g : R \to R$ be the linear function defined by $g(x) = 5x + 15$.
   Find the x - intercept and the y - intercept of g , then graph the function g .

S2.3 : Some well-known Functions and their Graphs

1) A function $f(x) = c$ where c is a fixed number is called a constant function .

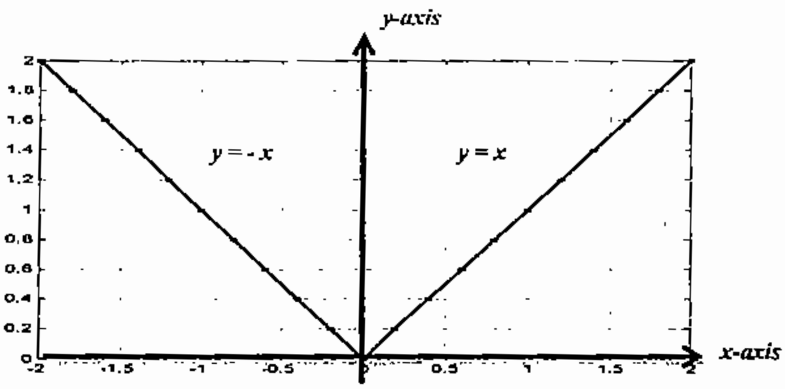
Example 2.3.1 : The function $y = f(x) = 1$ is a constant function and its graph is



2) The absolute value function $y = f(x) = |x|$ is defined by the formula

$$y = f(x) = |x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

and its graph is

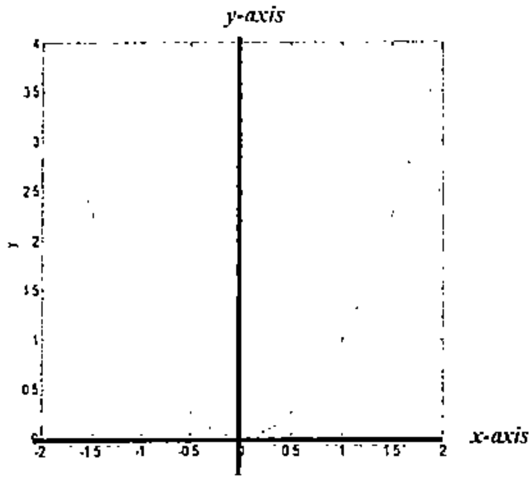


Remember that $|x| = \sqrt{x^2}$.

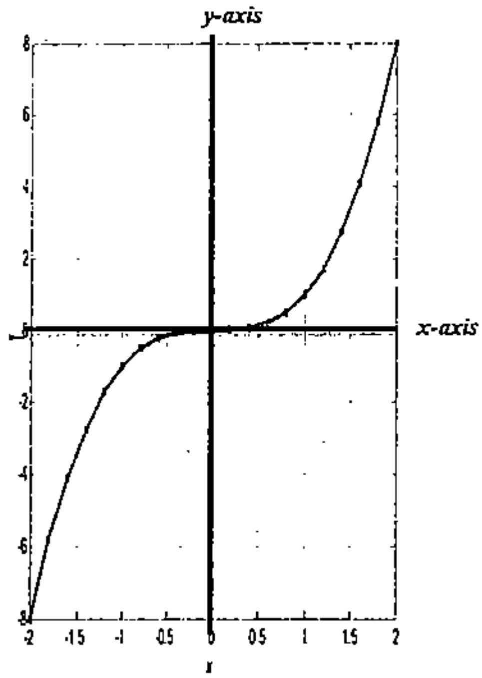
3) A function $y = f(x) = x^r$ where r is a real number is called a power function .

Example 2.3.2 :

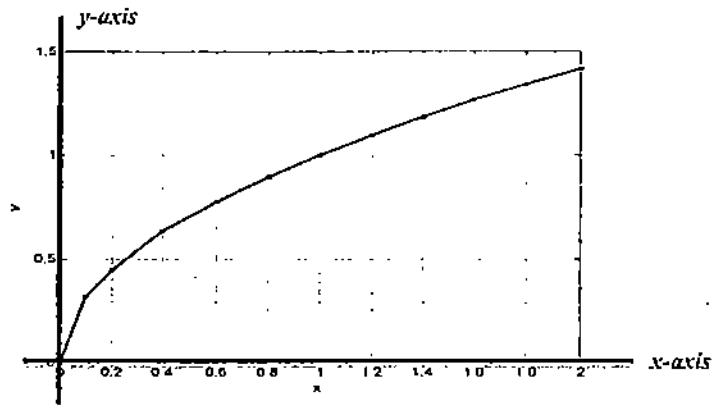
The function $y = f(x) = x^2$ is a power function (which is also a quadratic function) and its graph is



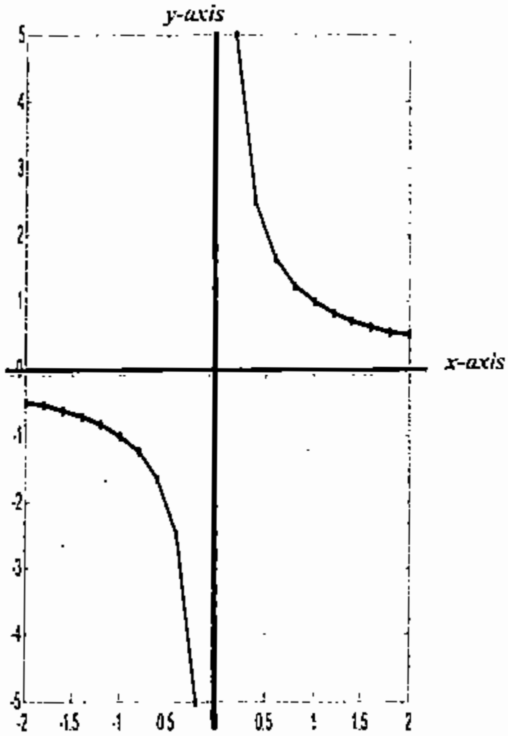
Example 2.3.3 : The function $y = f(x) = x^3$ is a power function and its graph is



Example 2.3.4 : The function $y = f(x) = \sqrt{x}$ is a power function and its graph is



Example 2.3.5 : The function $y = f(x) = \frac{1}{x}$ is a power function and its graph is



4) Let a be a positive real number other than 1 . The function $y = f(x) = a^x$ is called the exponential function with base a .

Example 2.3.6 : Graph the exponential function $y = 2^x$

Answer : To draw the graph of $y = 2^x$, we can make use of a table give values for x and find the corresponding values for y

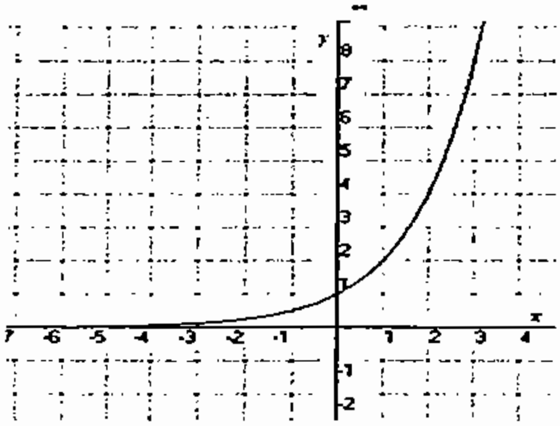
$x = 0$ gives $y = 2^0 = 1$,

$x = 1$ gives $y = 2^1 = 2$,

$x = -1$ gives $y = 2^{-1} = \frac{1}{2}$.

Following the process we make the table

x	-4	-3	-2	-1	0	1	2	3	4
2^x	0.0625	0.125	0.25	0.5	1	2	4	8	16



Example 2.3.7 : The function $y = 5^x$ is an exponential function and its graph is

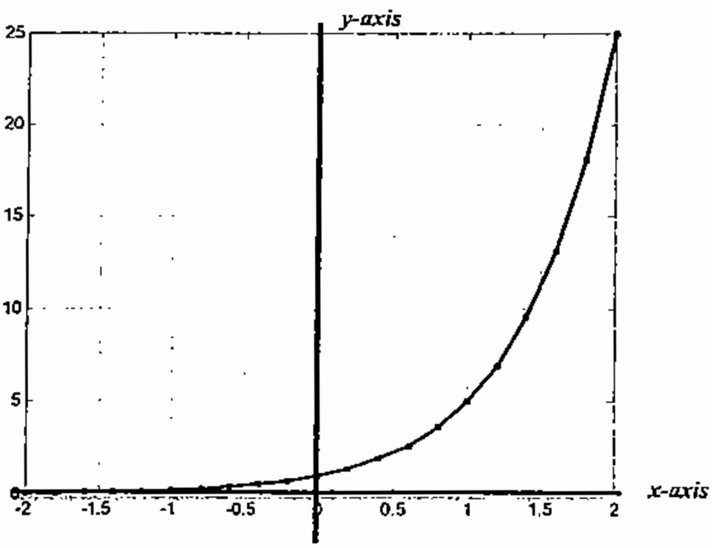
Answer :

$x = 0$ gives $y = 5^0 = 1$,

$x = 1$ gives $y = 5^1 = 5$,

$x = -1$ gives $y = 5^{-1} = 0.2$

x	-2	-1	0	1	2
5^x	0.04	0.2	1	5	25



Exercise 2.3.8 : Graph the exponential function $y = 10^x$.

The properties of exponential function and their graph

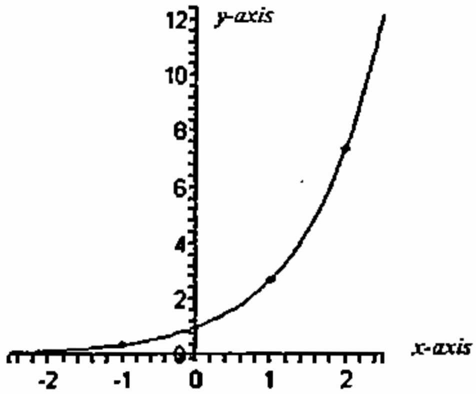
- The domain is R (set of real numbers) .
- The range is R^+ (set of positive real numbers) .
- The graph is always continuous (no break in the graph) .

Rules of Exponents : If $a > 0$ and $b > 0$, the following rules of exponent should be hold for all real numbers x and y :

1. $a^x \times a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $a^0 = 1$
4. $\frac{1}{a^x} = a^{-x}$
5. $(a^x)^y = (a^y)^x = a^{xy}$
6. $(ab)^x = a^x b^x$
7. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

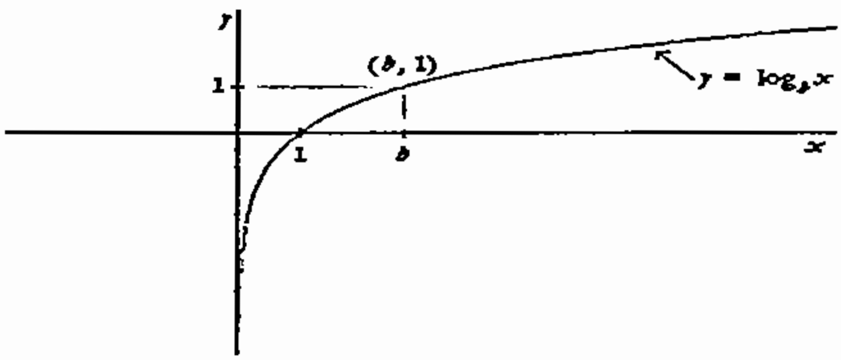
5) The function $y = e^x$ is called the natural exponential function whose base is $e \cong 2.718281828$, and its graph is

x	-2	-1	0	1	2
e^x	0.1353	0.3679	1	2.718	7.389



Remark : Graph of e^x and e^{-x} are reflections of each other .

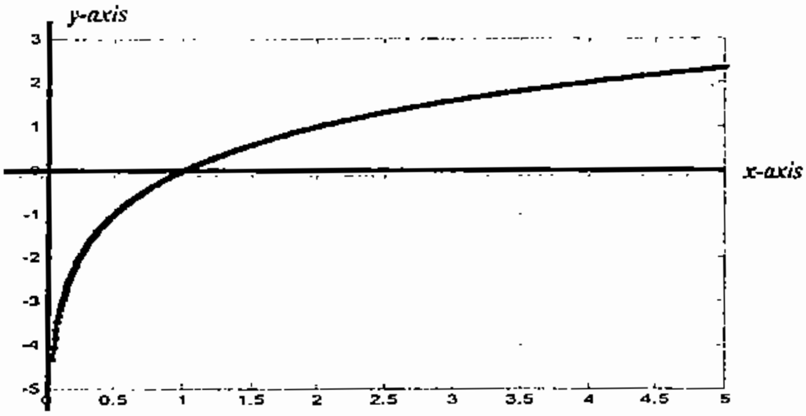
6) The function $y = \log_b x$ is called the logarithm function with base b where b is a positive number $\neq 1$; and $x > 0$, and the graph of $y = \log_b x$ where b is greater than 1 is the following graph



Remark : $y = \log_b x$ means that $x = b^y$.

Example 2.3.9 : The function $y = \log_2 x$ is a logarithm function with base 2 and its graph is

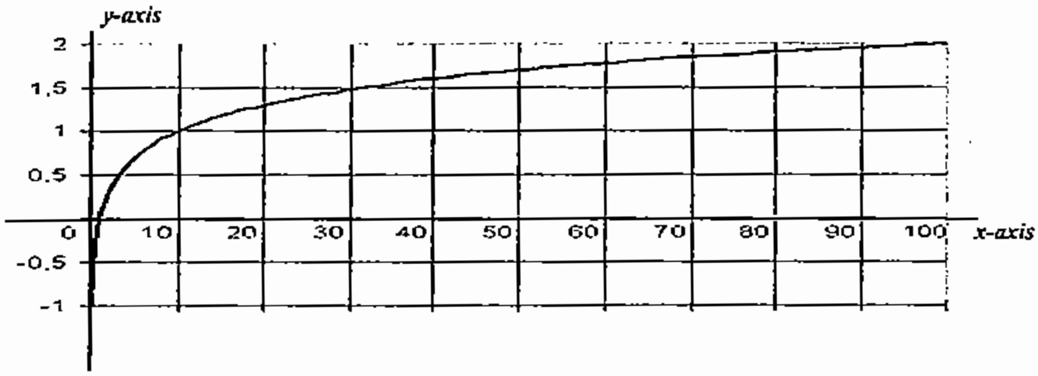
x	0.25	0.5	1	2	4
$y = \log_2 x$	-2	-1	0	1	2



Example 2.3.10 : Draw the graph of $\log_{10} x$.

Answer :

x	0.5	1	5	10	15	20	50	100
$y = \log_{10} x$	-0.301	0	0.699	1	1.176	1.301	1.699	2



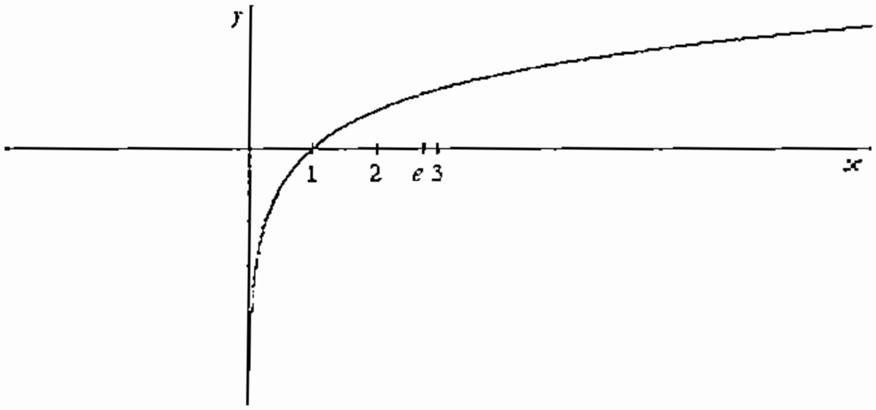
Rules of logarithm : For $x > 0$ and $y > 0$, and b is a positive number $\neq 1$ we have the following rules :

1. $\log_b xy = \log_b x + \log_b y$
2. $\log_b \frac{x}{y} = \log_b x - \log_b y$
3. $\log_b x^y = y \cdot \log_b x$
4. $\log_b a = \frac{\log_c a}{\log_c b}$, where c can be any base .

Remarks :

- The logarithm of any number to the base of the same number will be 1 ($\log_b b = 1$, $\log_5 5 = 1$ etc ...).
- Logarithm of 1 to any base is 0 ($\log_b 1 = 0$, $\log_3 1 = 0$ etc ...).
- The logarithm function is defined only for positive numbers .
- The domain of the logarithm function is R^+ .
- The range of the logarithm function is R .

7) The logarithm function with base e is called the natural logarithm function and will be denoted by $y = \ln x$ (i.e. $y = \log_e x = \ln x$) and its graph is



Remarks :

- $\ln e = 1$ (since $\ln e = \log_e e$)
- $\ln 1 = 0$

Exercise 2.3.12 : Draw the graph for the following logarithmic functions:

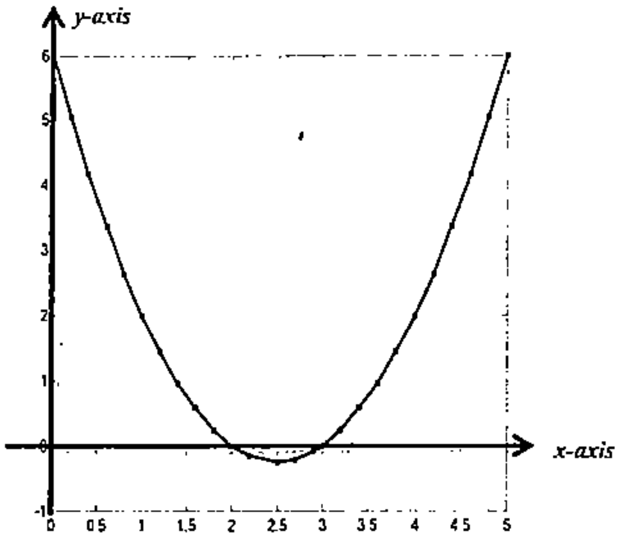
1. $\log_5 x$
2. $\log_8 x$
3. $\log_3 x$

8) A polynomial function is defined as

$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where

a_0 , a_1 , ... , a_{n-1} , a_n are constants .

Example 2.3.13 : The function $y = x^2 - 5x + 6$ is a polynomial function .



Algebra of Functions

Definition: The sum , difference , product , and quotient of the functions f and g are the functions defined by

$(f+g)(x) = f(x) + g(x)$ sum function

$(f-g)(x) = f(x) - g(x)$ difference function

$(f.g)(x) = f(x).g(x)$ product function

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $g(x) \neq 0$ quotient function

The domain of each function is the intersection of the domains of f and g , with the exception that the values of x where $g(x) = 0$ must be excluded from the domain of the quotient function .

Definition: Let f and g be functions , then $f \circ g$ is called the composite of g and f and is defined by the equation $(f \circ g)(x) = f(g(x))$.

The domain of $f \circ g$ is the set
$\mathbf{D} = \{ x \in \text{domain } g : g(x) \in \text{domain } f \}$.

Example 2.3.14 : Let f and g be the functions defined by $f(x) = x - 7$ and $g(x) = x^2 + 5$. Find the functions $f + g$, $f - g$, $f.g$, $\frac{g}{f}$, $f \circ g$, $g \circ f$ and find their domains .

Solution :

$(f+g)(x)=f(x)+g(x) = x-7 + x^2+5 = x^2+x-2$

$(f-g)(x)=f(x)-g(x) = x-7 - x^2-5 = -x^2+x-12$

$(f.g)(x)=f(x).g(x) = (x-7).(x^2+5) = x^3-7x^2+5x-35$

$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{x^2+5}{x-7}$

$(f\circ g)(x)=f(g(x)) = f(x^2+5) = x^2+5-7 = x^2-2$

$(g\circ f)(x)=g(f(x))= g(x-7) = (x-7)^2+5$

$= x^2-14x+49+5 = x^2-14x+54$

The domain of f $= \mathrm{R}$

The domain of g $= \mathrm{R}$

The intersection of the domains of f and g is R

Thus the domain of each of the functions $f+g$, $f-g$, $f.g$, $f\circ g$, and $g\circ f$ is R .

The domain of $\frac{g}{f}$ $= \mathrm{R}-\{7\}$.

Remark : The domain of any polynomial function is R .

Example 2.3.15 : Let f and g be the functions defined by $f(x)=x+5$ and $g(x) = x^2-3$, Find $f\circ g(x)$, $g\circ f(x)$, $f\circ g(3)$ and $g\circ f(3)$.

Solution: $f\circ g(x) = f(g(x)) = f(x^2-3)$

$= x^2-3+5$

$= x^2+2$

$g\circ f(x) = g(f(x)) = g(x+5)$

$= (x+5)^2-3$

$= x^2+10x+25-3$

$= x^2+10x+22$

$f \circ g(3) = (3)^2 + 2 = 9 + 2 = 11$

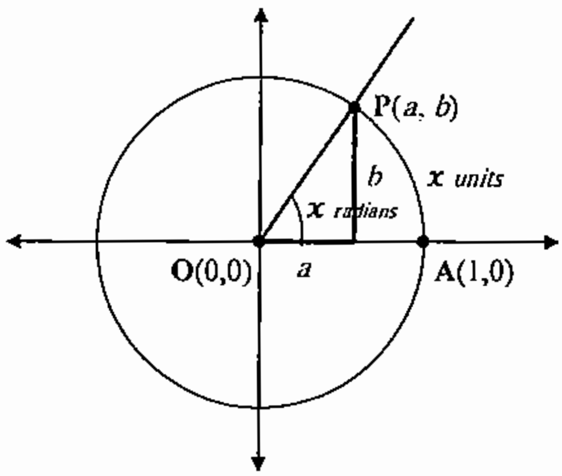
$g \circ f(3) = (3)^2 + 10(3) + 22 = 9 + 30 + 22 = 61$

Exersice 2.3.16 : Let f and g be the functions defined by $f(x) = x - 4$ and $g(x) = \sqrt{x}$. Find the functions $f+g$, $f-g$, $f \cdot g$, $\frac{f}{g}$ and find their domains .

S 2.4 : Unit Circle and Basic Trigonometric Functions

Definition 1: Let x be any real number and let U be the unit circle with equation $a^2 + b^2 = 1$ (the centre of the circle U is the point O (0,0) ; and the radius of the circle U equals 1) . Start from the point A(1,0) on U and proceed counterclockwise if x is positive and clockwise if x is negative around the unit circle U until an arc length of $|x|$ has been covered . Let P(a,b) be the point at the terminal end of the arc . The measurement of the angle AOP is x radians .

If x radians = t° (degrees) , then the following six trigonometric functions of x are defined in terms of the coordinates of the circular point P(a,b) :



1) $y = \sin x = b = \sin(x \text{ radians}) = \sin(t \text{ degrees}) = \sin t^\circ$

2) $y = \cos x = a = \cos(x \text{ radians}) = \cos(t \text{ degrees}) = \cos t^\circ$

3) $y = \tan x = \frac{b}{a} \quad (a \neq 0)$

$= \tan(x \text{ radians}) = \tan(t \text{ degrees}) = \tan t^\circ$

4) $y = \cot x = \frac{a}{b} \quad (b \neq 0)$

$= \cot(x \text{ radians}) = \cot(t \text{ degrees}) = \cot t^\circ$

5) $y = \sec x = \frac{1}{a} \qquad (a \neq 0)$

$= \sec (x \text{ radians}) = \sec (t \text{ degrees}) = \sec t^{\circ}$

6) $y = \csc x = \frac{1}{b} \qquad (b \neq 0)$

$= \csc (x \text{ radians}) = \csc (t \text{ degrees}) = \csc t^{\circ}$

<u>**Remark 1:**</u> Definition 1 uses the standard function notation , $y = f(x)$, with f replaced by the name of a particular trigonometric function . For example , $y = \cos x$ actually means $y = \cos(x)$ and
$\cos t^{\circ}$ actually means $\cos(t^{\circ})$.

<u>**Remark 2:**</u> Remember that $t^{\circ} = t \times \frac{\pi}{180}$ *radians* and
$x \text{ radians} = (x \times \frac{180}{\pi})^{\circ}$.

<u>**Theorem 1:**</u>

For any real number x we have the following trigonometric identities :

1) $\csc x = \frac{1}{\sin x}$.
2) $\sec x = \frac{1}{\cos x}$.
3) $\cot x = \frac{1}{\tan x}$.
4) $\tan x = \frac{\sin x}{\cos x}$.
5) $\cot x = \frac{\cos x}{\sin x}$.
6) $\sin(-x) = -\sin(x)$.
7) $\cos(-x) = \cos(x)$.
8) $\tan(-x) = -\tan(x)$.
9) $\cot(-x) = -\cot(x)$.
10) $\sin^2 x + \cos^2 x = 1$.
11) $\sec^2 x = \tan^2 x + 1$.
12) $\csc^2 x = \cot^2 x + 1$.

S 2.5: Graphs of Sine and Cosine Functions

2.5.1: Table for values of $\sin x$, $\cos x$, and $\tan x$ for selected values of x

Values of x						
	Degrees	0	30	45	60	90
	Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$		0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$		1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan x$		0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

Values of x						
	Degrees	120	135	150	180	270
	Radians	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$
$\sin x$		$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\cos x$		$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0
$\tan x$		$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	Undefined

Definition: A function f is periodic if there exists a positive real number p such that $f(x)=f(x+p)$ for all x in the domain of f.
The smallest such positive number p is the period of f.

Remarks :

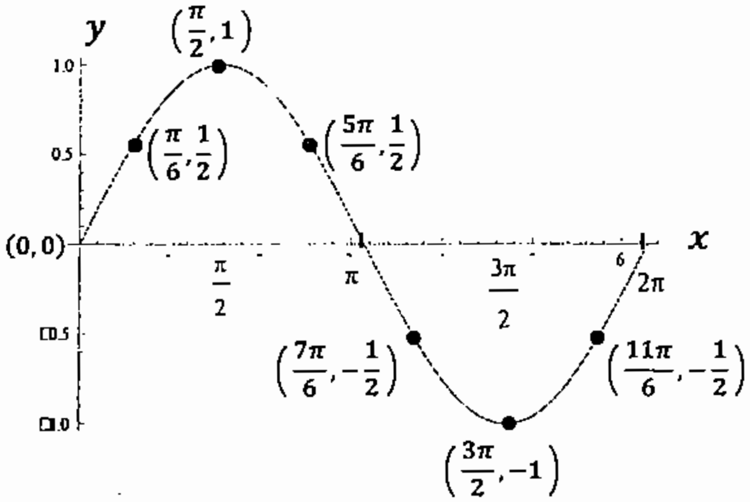
1) The functions $\sin x$, $\cos x$, $\sec x$, and $\csc x$ are periodic functions with period 2π .

2) The functions $\tan x$ and $\cot x$ are periodic functions with period π .

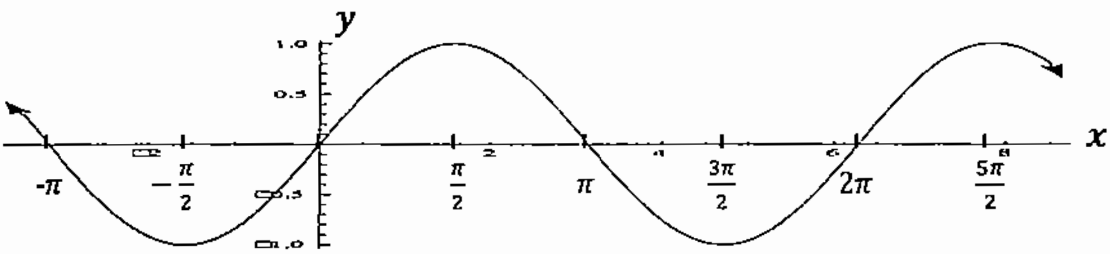
2.5.2: The Graph of $\sin x$

The graph of the function $y = \sin x$ is the line passing through all the points $(x, \sin x)$ on the xy-plane .

The graph of the function $y = \sin x$ for the interval $[0, 2\pi]$ is the line passing through the points $(0, 0)$, $(\frac{\pi}{6}, \frac{1}{2})$, $(\frac{\pi}{2}, 1)$, $(\frac{5\pi}{6}, \frac{1}{2})$, $(\pi, 0)$, $(\frac{7\pi}{6}, -\frac{1}{2})$, $(\frac{3\pi}{2}, -1)$, $(\frac{11\pi}{6}, -\frac{1}{2})$, and $(2\pi, 0)$ which is shown in the following figure



The graph of the function $y = \sin x$ is shown in the following figure



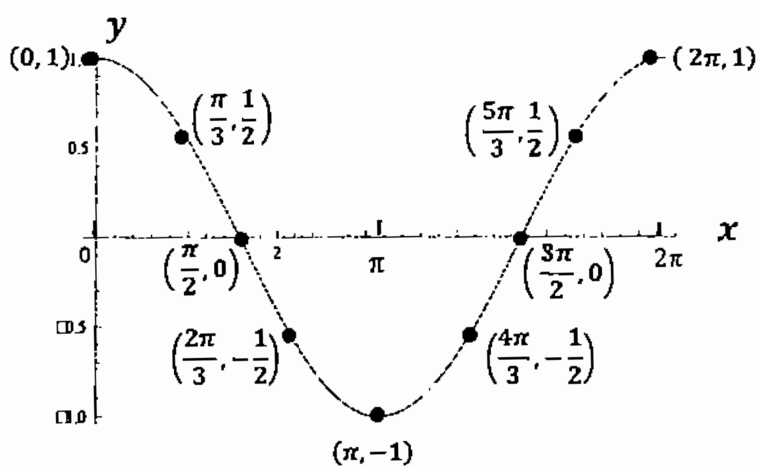
The period of the function $y = \sin x$ is 2π . The domain of the function $y = \sin x$ is the set of all real numbers $\mathbb{R}$.

The range of the function $y = \sin x$ is the interval $[-1, 1]$.

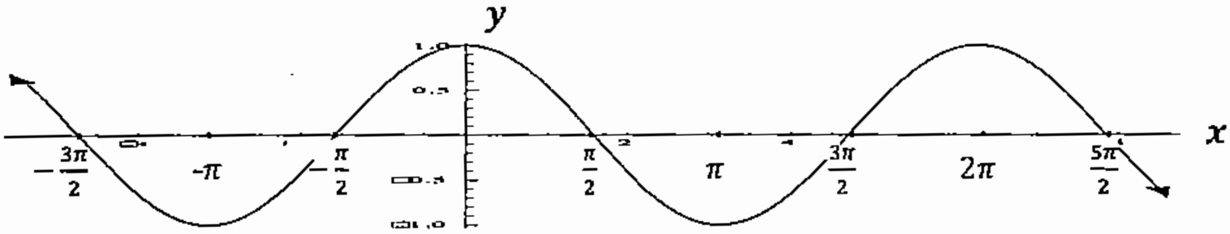
2.5.3: The Graph of $\cos x$

The graph of the function $y = \cos x$ is the line passing through all the points $(x, \cos x)$ on the xy-plane.

The graph of the function $y = \cos x$ for the interval $[0, 2\pi]$ is the line Passing through the points $(0, 1)$, $(\frac{\pi}{3}, \frac{1}{2})$, $(\frac{\pi}{2}, 0)$, $(\frac{2\pi}{3}, -\frac{1}{2})$, $(\pi, -1)$, $(\frac{4\pi}{3}, -\frac{1}{2})$, $(\frac{3\pi}{2}, 0)$, $(\frac{5\pi}{3}, \frac{1}{2})$, and $(2\pi, 1)$ which is shown in the following figure



The graph of the function $y = \cos x$ is shown in the following figure

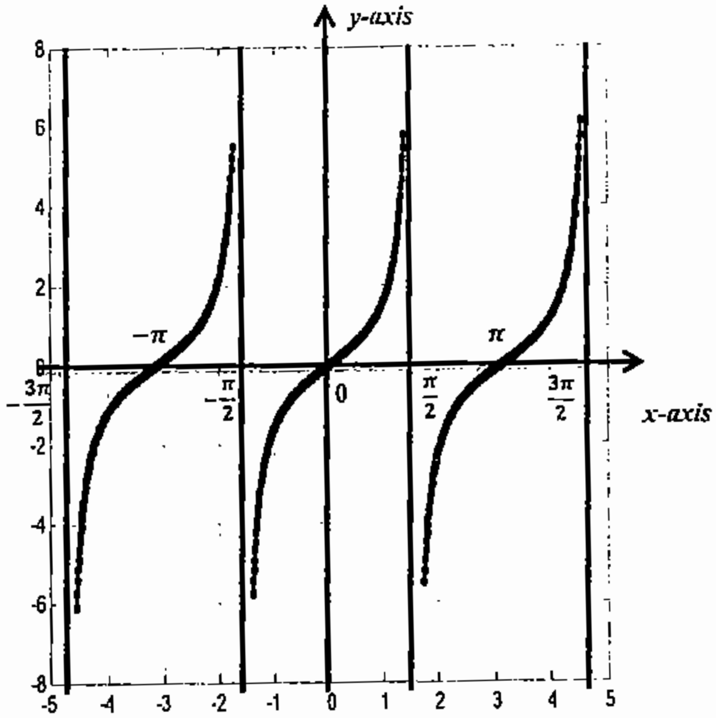


The period of the function $y = \cos x$ is 2π.
The domain of the function $y = \cos x$ is the set of all real numbers R.
The range of the function $y = \cos x$ is the interval $[-1, 1]$.

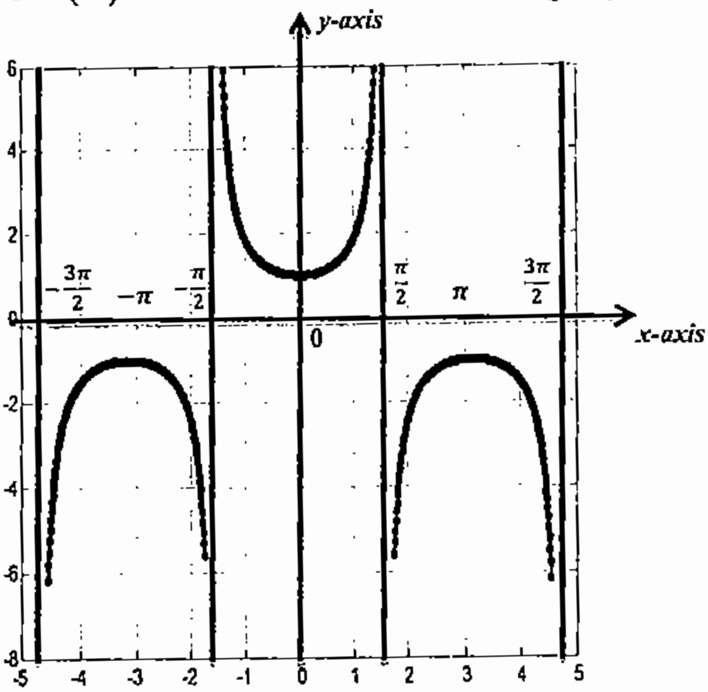
2.5.4: The Graphs of $\tan x$ and $\sec x$

The graph of the function $y = \tan(x)$ is the line passing through all the points $(x, \tan x)$ on the xy-plane .

The graph of $y = \tan(x)$ is shown in the following figure



The graph of $y = \sec(x)$ is shown in the following figure



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<u>Exercise:</u> Draw the graph of the following trigonometric functions :

1) $y = \csc(x)$

2) $y = \cot(x)$

CH3 : Limits , Continuity and Differentiation

S3.1 : Limits and Continuity

Remark 3.1.1: If the values of a function $y = f(x)$ can be made as close as
we like to a fixed number L by taking x close to x_0 (but not equal to x_0) we say that L is the limit of f as x approaches x_0 , and we write it as

$$\lim_{x \to x_0} f(x) = \mathrm{L}$$

Also we can say that the limit of f as x approaches x_0 equals L .

Definition 3.1.2 :

Let f be a function defined on the set $(x_0 - p, x_0) \cup (x_0, x_0 + p)$, with $p > 0$. Then

$$\lim_{x \to x_0} f(x) = \mathrm{L}$$

iff for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

if $0 < |x - x_0| < \delta$ **then** $|f(x) - \mathrm{L}| < \varepsilon$.

Theorem 1 :

1) $\lim_{x \to x_0} x = x_0$
2) $\lim_{x \to x_0} \mathrm{k} = \mathrm{k}$

Theorem 2 : If $\lim_{x \to x_0} f(x) = \mathrm{L}_1$ and $\lim_{x \to x_0} g(x) = \mathrm{L}_2$, then

1) $\lim_{x \to x_0} [f(x) + g(x)] = \mathrm{L}_1 + \mathrm{L}_2$

2) $\lim_{x \to x_0} [f(x) - g(x)] = \mathrm{L}_1 - \mathrm{L}_2$

3) $\lim_{x \to x_0} [f(x) \, . \, g(x)] = \mathrm{L}_1 \, . \, \mathrm{L}_2$

4) $\lim_{x \to x_0} [k \, . \, f(x)] = \mathrm{k} \, . \, \mathrm{L}_1$

5) $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\mathrm{L}_1}{\mathrm{L}_2}$ if $\mathrm{L}_2 \neq 0$.

Example 3.1.3 : Find each of the following :

1. $\lim\limits_{x \to -2} 7$

2. $\lim\limits_{x \to 1} x(3-x)$

3. $\lim\limits_{x \to 3} (x^2+2x-1)$

4. $\lim\limits_{x \to 2} \dfrac{x-2}{x^2-5x+6}$

5. $\lim\limits_{x \to 0} \dfrac{x^2-5x}{x}$

Solution :

1. $\lim\limits_{x \to -2} 7 = 7$

2. $\lim\limits_{x \to 1} x(3-x) = 1(3-1) = 2$

3. $\lim\limits_{x \to 3} (x^2+2x-1) = (3)^2+2(3)-1 = 9+6-1 = 14$

4. $\lim\limits_{x \to 2} \dfrac{x-2}{x^2-5x+6} = \lim\limits_{x \to 2} \dfrac{x-2}{(x-3)(x-2)} = \lim\limits_{x \to 2} \dfrac{1}{(x-3)} = \dfrac{1}{2-3} = -1$

5. $\lim\limits_{x \to 0} \dfrac{x^2-5x}{x} = \lim\limits_{x \to 0} \dfrac{x(x-5)}{x} = \lim\limits_{x \to 0} (x-5) = 0-5 = -5$

Theorem 3 :

1) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

2) $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x} = 0$

Example 3.1.4 : Find each of the following :

1. $\lim\limits_{x \to 0} \dfrac{\sin 4x}{\sin 5x}$

2. $\lim\limits_{x \to 0} \dfrac{3x}{\sin 2x}$

3. $\lim_{x \to 0} \frac{\tan x}{x}$

Solution :

1. $\lim_{x \to 0} \frac{\sin 4x}{\sin 5x} = \lim_{x \to 0} \frac{4x \cdot \frac{\sin 4x}{4x}}{5x \cdot \frac{\sin 5x}{5x}} = \frac{4}{5}$

2. $\lim_{x \to 0} \frac{3x}{\sin 2x} = \lim_{x \to 0} \frac{3x}{2x \cdot \frac{\sin 2x}{2x}} = \frac{3}{2}$

3. $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{\frac{x}{1}} = \lim_{x \to 0} \left(\frac{\sin x}{\cos x} \cdot \frac{1}{x} \right) = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right)$

$= 1 \times 1 \quad = 1$

Exercise 3.1.5 : Find each of the following :

1. $\lim_{x \to 0} \frac{1 - \cos x}{x + \sin x}$
2. $\lim_{x \to \infty} \left(1 + \cos \frac{1}{x}\right)$
3. $\lim_{x \to 0} \frac{\sin 2x}{2x^2 + x}$
4. $\lim_{y \to 0} \frac{\tan 2y}{3y}$
5. $\lim_{y \to \infty} \frac{y^4}{y^4 - 7y^3 + 3y^2 + 9}$

Definition 3.1.6 : A function $f(x)$ is said to be continuous at x_0 if

1) f is defined at x_0 (i.e. $f(x_0) = L$ where $L \in R$).

2) $\lim_{x \to x_0} f(x)$ exists

3) $\lim_{x \to x_0} f(x) = f(x_0) = L$

Example 3.1.7 : Let $f(x) - \begin{cases} x^2 & x \leq 1 \\ 3-2x & x > 1 \end{cases}$

Is f continuous at $x = 1$.

Solution :

1) $f(1) = 1^2 = 1$

2) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = 1^2 = 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (3 - 2x) = 3 - 2(1) = 1$

since $\lim\limits_{x \to 1^-} f(x) = 1 = \lim\limits_{x \to 1^+} f(x)$

Therefore $\lim\limits_{x \to 1} f(x)$ exists and $\lim\limits_{x \to 1} f(x) = 1$

3) $\lim\limits_{x \to 1} f(x) = 1 = f(1)$

Therefore f is continuous at $x = 1$

Example 3.1.8 : Let $f(x) = \begin{cases} 2x+1 & if \ x < -2 \\ x^2 - 2 & if \ x \geq -2 \end{cases}$

Is f continuous at $x = -2$.

Solution :

1) $f(-2) = (-2)^2 - 2 = 4 - 2 = 2$

2) $\lim\limits_{x \to (-2)^-} f(x) = \lim\limits_{x \to (-2)^-} (2x + 1) = 2(-2) + 1 = -4 + 1 = -3$

$\lim\limits_{x \to (-2)^+} f(x) = \lim\limits_{x \to (-2)^+} (x^2 - 2) = (-2)^2 - 2 = 4 - 2 = 2$

since $\lim\limits_{x \to (-2)^-} f(x) \neq \lim\limits_{x \to (-2)^+} f(x)$

Therefore $\lim\limits_{x \to (-2)} f(x)$ does not exists

Thus f is not continuous at $x = -2$.

Exercise 3.1.9 :

Let $f(x) = \begin{cases} \frac{x^2-2x-8}{x+2} & if \ x \neq -2 \\ -3 & if \ x = -2 \end{cases}$

Is f continuous at $x = -2$.

S3.2 : Differentiation

Definition of Derivative , Rules of Differentiation

Definition 3.2.1:

Let $y = f(x)$ be a function and let the variable x receive a certain increment Δx. Then the function y will receive a certain increment Δy. Thus for the value of x we have $y = f(x)$ and for the value of $x + \Delta x$, we have $y + \Delta y = f(x + \Delta x)$.

Thus the increment Δy is given by :

$$\Delta y = f(x + \Delta x) - f(x)$$

Remark 3.2.2 : **Δ is an abbreviation of difference (in x , y) and is not a factor .**

Forming the ratio of the increment of the function y to the increment of the variable x , we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is called the average rate of change of the function $y = f(x)$ with respect to the variable x . $\frac{\Delta y}{\Delta x}$ is also called the difference quotient of the function $y = f(x)$. If the limit of this ratio as Δx approaches zero exists, that is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exist, then the function is called differentiable and the limit $(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x})$

is called the first derivative of the function $y = f(x)$ with respect to

the variable x, which is denoted by $f'(x)$, y', $\frac{dy}{dx}$, $\frac{d}{dx}y$, $\frac{d}{dx}f(x)$.

Differentiation Rules:

Let $f(x)$ and $g(x)$ be two differentiable functions (in the interval under consideration) , then

RULE 1 Constant Multiple Rule

If $f(x)$ is a differentiable function of x, and c is a constant , then

$$\frac{d}{dx}(\, c f(x)\,) = c\,\frac{d}{dx} f(x)\,.$$

RULE 2 Derivative of the Sum

If $f(x)$ and $g(x)$ are differentiable functions of x, then their sum $f(x) + g(x)$ is differentiable , and

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

RULE 3 Derivative of the Difference

If $f(x)$ and $g(x)$ are differentiable functions of x, then their difference $f(x) - g(x)$ is differentiable , and

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

RULE 4 Derivative of the Product

If $f(x)$ and $g(x)$ are differentiable functions of x, then their product $f(x).\, g(x)$ is differentiable , and

$$\frac{d}{dx}(f(x).g(x)) = f(x).\frac{d}{dx} g(x) + g(x).\frac{d}{dx} f(x)$$

RULE 5 Derivative of the Quotient

If $f(x)$ and $g(x)$ are differentiable functions of x and $g(x) \neq 0$, then the quotient $\dfrac{f(x)}{g(x)}$ is differentiable , and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \frac{d}{dx} f(x) - f(x) \cdot \frac{d}{dx} g(x)}{g(x)^2}$$

Derivatives of Some Special Functions and the Chain Rule:

1) Derivatives of Some Algebraic Functions:

1) Derivative of a Constant Function

If $f(x) = c$, then $\frac{d}{dx} f(x) = \frac{d}{dx} c = 0$

Example 3.2.3 : If $f(x)=12$, then $\frac{d}{dx} f(x) = \frac{d}{dx}(12) = 0$.

2) Derivatives of a Power Functions

$$\frac{d}{dx} x^n = n x^{n-1} , \quad n \in Q$$

provided that $x \neq 0$ when n is negative .

Example 3.2.4 : Find f' for each of the following functions :

(i) $f(x) = x$, (ii) $f(x) = x^2$, (iii) $f(x) = x^{-3}$, (iv) $f(x) = x^{0.3}$

Solution:

(i) $f'(x) = x^{1-1} = x^0 = 1$

(ii) $f'(x) = 2x^{2-1} = 2x$

(iii) $f'(x) = -3x^{-3-1} = -3x^{-4}$

(iv) $f'(x) = 0.3x^{0.3-1} = 0.3x^{-0.7}$

Example 3.2.5 : Find f' for each of the following functions :

$(i)\ f(x)=\frac{1}{2}x$, $(ii)\ f(x)=9x^2$, $(iii)\ f(x)=4x^{-3}$, $(iv)\ f(x)=x^{2.5}$,

Solution:

$(i)\ f'(x)=\frac{1}{2}(1)=\frac{1}{2}$

$(ii)\ f'(x)=9(2x^{2-1}) = 18x$

$(iii)\ f'(x)=4((-3)x^{-3-1}) = -12x^{-4}$

$(iv)\ f'(x)=2.5x^{2.5-1}=2.5x^{1.5}$

Example 3.2.6 : Find f' for each of the following functions :

$(i)\ f(x)=x^2+5x^{-3}$, $(ii)\ f(x)=x^4-\frac{3}{5}x^2+7x-14$

Solution:

$(i)\ f'(x)=2x-15x^{-4}$

$(ii)\ f'(x)=4x^3-\frac{3(2)}{5}x+7-0=4x^3-\frac{6}{5}x+7$

Example 3.2.7 : Find f' for the function $f(x)=2x(3x^5+\frac{3}{x})$

Solution:

$$f'(x)=2x\,(15x^4-\frac{3}{x^2})+(3x^5+\frac{3}{x}).2$$

$$=30x^5-\frac{6}{x}+6x^5+\frac{6}{x}=36x^5$$

Example 3.2.8 : Find f' for the function $f(x)=\frac{2x-1}{3x+1}$

Solution:

$$f'(x)=\frac{(3x+1).2-(2x-1).3}{(3x+1)^2}$$

$$=\frac{6x+2-6x+3}{(3x+1)^2}=\frac{5}{(3x+1)^2}$$

The derivative of the cosine function is the negative of the sine function :

$$\frac{d}{dx}(\cos x) = -\sin x$$

Example 3.2.10 : Find $f'(x)$ for the function $f(x) = 3x^2 + 2\cos x$

Solution: $f'(x) = 6x - 2\sin x$

Example 3.2.11 : Find y' for each of the following functions :

(i) $y = \sin x - \cos x$ (ii) $y = 2\sin x \cos x$ (iii) $y = \dfrac{3\sin x}{\cos x + 1}$

Solution:

(i) $y' = \cos x + \sin x$

(ii) $y' = 2\sin x \,.\, (-\sin x) + \cos x \,.\, (2\cos x) = -2\sin^2 x + 2\cos^2 x$

$$(iii)\ y' = \frac{(\cos x + 1)\,.\,(3\cos x) - (3\sin x)\,.\,(-\sin x)}{(\cos x + 1)^2}$$

$$= \frac{3\cos^2 x + 3\cos x + 3\sin^2 x}{(\cos x + 1)^2}$$

The derivative of other trigonometric functions :

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 3.2.12 : Find y' for each of the following functions :

(i) $y = \tan x + \sec x$ (ii) $y = 5\cot x \csc x$

Solution:

$(i)\ \ y' = \sec^2 x + \sec x \tan x$

$(ii)\ \ y' = 5\cot x \,.\, (-\csc x \cot x\,) + \csc x \,.\, (-5\csc^2 x\,)$
$\quad = -5\csc x \cot^2 x - 5\csc^3 x$

Derivative of Logarithmic Function:

> **The derivative of the natural logarithmic function is:**
>
> $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Example 3.2.13 : Find y' for each of the following functions :

$(i)\ \ y = 4x^3 \ln x \qquad (ii)\ \ y = \dfrac{2\ln x}{9x+1}$

Solution:

(i) $y' = 4x^3\left(\frac{1}{x}\right) + \ln x\,(12x^2) = 4x^2 + 12x^2 \ln x$

$(ii)\ \ y' = \dfrac{(9x+1)\left(\frac{2}{x}\right) - (2\ln x)(9)}{(9x+1)^2} = \dfrac{18 + \frac{2}{x} - 18\ln x}{(9x+1)^2}$

Derivative of Exponential Function :

> **The derivative of the exponential functions are:**
>
> $$\frac{d}{dx}a^x = a^x \ln a \quad \text{and} \quad \frac{d}{dx}e^x = e^x$$

Example 3.2.14 : Find f' for the function $f(x) = 5x^7 e^x + 4\,e^x$.

Solution: $f'(x) = 5x^7 e^x + e^x\,(35x^6) + 4\,e^x = 5x^7 e^x + 35x^6 e^x + 4\,e^x$

Implicit Differentiation (Derivative of Composite Functions) :

Chain Rule :

Let $y = f(u)$, $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Example 3.2.15 : Let $y = 6u^3 + 5u$, $u = \ln x$, find $\frac{dy}{dx}$.

Solution:

$\frac{dy}{du} = 18u^2 + 5$, $\frac{du}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (18u^2 + 5)(\frac{1}{x}) = \left(18(\ln x)^2 + 5\right)(\frac{1}{x})$$

$$= \frac{18}{x}(\ln x)^2 + \frac{5}{x}$$

Example 3.2.16 : Find $\frac{dy}{dx}$ for each of the following functions :

(i) $y = (x + 4x^3)^6$, (ii) $y = \ln(x^2 + 3)$, (iii) $y = \tan^3 x$.

Solution:

(i) let $u = x + 4x^3$, then $y = u^6$.

Thus $\frac{dy}{du} = 6u^5$ and $\frac{du}{dx} = 1 + 12x^2$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6u^5(1 + 12x^2) = 6(x + 4x^3)^5(1 + 12x^2) .$$

(ii) let $u = x^2 + 3$, then $y = \ln u$.

Thus $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 2x$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u}(2x) = \frac{2x}{x^2 + 3}$$

(iii) let $u = \tan x$, then $y = u^3$.

Thus $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = \sec^2 x$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 (\sec^2 x) = 3\tan^2 x \sec^2 x$$

In examples (3.2.15 and 3.2.16) we use the Chain rule to get the derivative of a composite function using substitutions, but also we can get the same results directly without substitutions, considering the following rules:

$$\frac{d}{dx}(f(x))^n = n\,(f(x))^{n-1} \, . \, f'(x),$$

$$\frac{d}{dx}(\ln f(x)) = \frac{1}{f(x)} \, . \, f'(x),$$

$$\frac{d}{dx}(e^{f(x)}) = e^{f(x)} \, . \, f'(x),$$

$$\frac{d}{dx}(\sin f(x)) = (\cos f(x)) \, . \, f'(x),$$

$$\frac{d}{dx}(\cos f(x)) = (-\sin f(x)) \, . \, f'(x),$$

$$\frac{d}{dx}(\tan f(x)) = (\sec^2 f(x)) \, . \, f'(x),$$

$$\frac{d}{dx}(\sec f(x)) = (\sec f(x) \, . \, \tan f(x)) \, . \, f'(x),$$

$$\frac{d}{dx}(\csc f(x)) = (-\csc f(x) \, . \, \cot f(x)) \, . \, f'(x),$$

$$\frac{d}{dx}(\cot f(x)) = (-\csc^2 f(x)) \, . \, f'(x).$$

<u>Example 3.2.17 :</u> Find $\frac{dy}{dx}$ for each of the following functions :

$(i)\ y=\sqrt{x^5+4x}$, $(ii)\ y=\ln(x^2+3x)$, $(iii)\ y=e^{3x}$.

<u>Solution:</u>

$(i)\ y=\sqrt{x^5+4x} = (x^5+4x)^{\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(x^5+4x)^{-\frac{1}{2}} \, . \, (5x^4+4) = \frac{5x^4+4}{2\sqrt{x^5+4x}} \, .$$

Solution:

$(i)\ y' = 20x^4 - 21x^2 + 3 \quad , \quad y'' = 80x^3 - 42x$

$(ii)\ y' = x^3\,(4e^{4x}) + e^{4x}\,(3x^2) = 4x^3 e^{4x} + 3x^2 e^{4x}$

$y'' = 4x^3\,(4e^{4x}) + e^{4x}\,(12x^2) + 3x^2\,(4e^{4x}) + e^{4x}\,(6x)$

$= 16x^3 e^{4x} + 12x^2 e^{4x} + 12x^2 e^{4x} + 6x\,e^{4x}$

$= 16x^3 e^{4x} + 24x^2 e^{4x} + 6x\,e^{4x}$

$(iii)\ y' = 2\cos x - 9\sin x$

$y'' = -2\sin x - 9\cos x$

Example 3.2.19 : Find y', y'', y''' and $y^{(4)}$ for each of the following functions :

$(i)\ y = x^6 + x^4 - 3x^3 \quad , \quad (ii)\ y = e^{2x} \quad , \quad (iii)\ y = \sin x \quad , \quad (iv)\ y = \cos x$

Solution:

$(i)\ y' = 6x^5 + 4x^3 - 9x^2 \quad , \quad y'' = 30x^4 + 12x^2 - 18x$

$, \quad y''' = 120x^3 + 24x - 18 \quad , \quad y^{(4)} = 360x^2 + 24 \quad .$

$(ii)\ y' = 2e^{2x} \quad , \quad y'' = 4e^{2x} \quad , \quad y''' = 8e^{2x} \quad , \quad y^{(4)} = 16e^{2x}$

$(iii)\ y' = \cos x \quad , \quad y'' = -\sin x \quad , \quad y''' = -\cos x \quad , \quad y^{(4)} = \sin x$

$(iv)\ y' = -\sin x \quad , \quad y'' = -\cos x \quad , \quad y''' = \sin x \quad , \quad y^{(4)} = \cos x$

S3.3 : L'Hopital Rule

Suppose that $f(x_0) = g(x_0) = 0$, and both $f'(x_0)$ and $g'(x_0)$ exist . Then $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{f'(x_0)}{g'(x_0)}$ if $g'(x_0) \neq 0$.

$(ii)\ y = \ln(x^2 + 3x)$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2 + 3x} \cdot (2x + 3) = \frac{2x+3}{x^2 + 3x} .$$

$(iii)\ y = e^{3x}$

$$\therefore \frac{dy}{dx} = e^{3x} \cdot 3 = 3\, e^{3x} .$$

Second Order Derivative and Derivatives of Higher Order:

When we differentiate a function $y = f(x)$ we get a new function y' (or $\frac{dy}{dx}$ or $f'(x)$ or $\frac{d}{dx}f$) which is the derivative of $y = f(x)$ (or the first derivative of $y = f(x)$) . Now if this derivative $y' = f'(x)$ is also a differentiable function , we can define the second derivative of $y = f(x)$ (or the second order derivative of $y = f(x)$) by differentiating y' (or $\frac{dy}{dx}$ or $f'(x)$ or $\frac{d}{dx}f$) , which is denoted by y'' (or $\frac{d^2y}{dx^2}$ or $f''(x)$ or $\frac{d^2}{dx^2}f$) .

Now if the second derivative $y'' = f''(x)$ is also a differentiable function , we can define the third derivative of $y = f(x)$ (or the third order derivative of $y = f(x)$) by differentiating y'' (or $\frac{d^2y}{dx^2}$ or $f''(x)$ or $\frac{d^2}{dx^2}f$) , which is denoted by y''' (or $\frac{d^3y}{dx^3}$ or $f'''(x)$ or $\frac{d^3}{dx^3}f$) . So long as we have differentiability , we can continue in this manner forming the fourth derivative of $y = f(x)$, which is denoted by $y^{(4)}$ (or $\frac{d^4y}{dx^4}$ or $f^{(4)}(x)$ or $\frac{d^4}{dx^4}f$) , and more generally the nth derivative of $y = f(x)$ is denoted by $y^{(n)}$ (or $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$ or $\frac{d^n}{dx^n}f$) .

<u>Example 3.2.18 :</u> Find y'' for each of the following functions :

$(i)\ y = 4x^5 - 7x^3 + 3x$, $(ii)\ y = x^3 e^{4x}$, $(iii)\ y = 2\sin x + 9\cos x$

$(ii)\ y = \ln(x^2 + 3x)$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2+3x} \cdot (2x+3) = \frac{2x+3}{x^2+3x} .$$

$(iii)\ y = e^{3x}$

$$\therefore \frac{dy}{dx} = e^{3x} \cdot 3 = 3e^{3x} .$$

Second Order Derivative and Derivatives of Higher Order:

When we differentiate a function $y = f(x)$ we get a new function y' (or $\frac{dy}{dx}$ or $f'(x)$ or $\frac{d}{dx}f$) which is the derivative of $y = f(x)$ (or the first derivative of $y = f(x)$) . Now if this derivative $y' = f'(x)$ is also a differentiable function , we can define the second derivative of $y = f(x)$ (or the second order derivative of $y = f(x)$) by differentiating y' (or $\frac{dy}{dx}$ or $f'(x)$ or $\frac{d}{dx}f$), which is denoted by y'' (or $\frac{d^2y}{dx^2}$ or $f''(x)$ or $\frac{d^2}{dx^2}f$) .

Now if the second derivative $y'' = f''(x)$ is also a differentiable function , we can define the third derivative of $y = f(x)$ (or the third order derivative of $y = f(x)$) by differentiating y'' (or $\frac{d^2y}{dx^2}$ or $f''(x)$ or $\frac{d^2}{dx^2}f$), which is denoted by y''' (or $\frac{d^3y}{dx^3}$ or $f'''(x)$ or $\frac{d^3}{dx^3}f$) . So long as we have differentiability , we can continue in this manner forming the fourth derivative of $y = f(x)$, which is denoted by $y^{(4)}$ (or $\frac{d^4y}{dx^4}$ or $f^{(4)}(x)$ or $\frac{d^4}{dx^4}f$) , and more generally the nth derivative of $y = f(x)$ is denoted by $y^{(n)}$ (or $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$ or $\frac{d^n}{dx^n}f$) .

Example 3.2.18 : Find y'' for each of the following functions :

$(i)\ y = 4x^5 - 7x^3 + 3x$, $(ii)\ y = x^3 e^{4x}$, $(iii)\ y = 2\sin x + 9\cos x$

Solution:

$(i)\ y' = 20x^4 - 21x^2 + 3 \quad , \quad y'' = 80x^3 - 42x$

$(ii)\ y' = x^3(4e^{4x}) + e^{4x}(3x^2) = 4x^3e^{4x} + 3x^2e^{4x}$

$y'' = 4x^3(4e^{4x}) + e^{4x}(12x^2) + 3x^2(4e^{4x}) + e^{4x}(6x)$

$= 16x^3e^{4x} + 12x^2e^{4x} + 12x^2e^{4x} + 6xe^{4x}$

$= 16x^3e^{4x} + 24x^2e^{4x} + 6xe^{4x}$

$(iii)\ y' = 2\cos x - 9\sin x$

$y'' = -2\sin x - 9\cos x$

Example 3.2.19 : **Find y' , y'' , y''' and $y^{(4)}$ for each of the following functions :**

$(i)\ y = x^6 + x^4 - 3x^3 \quad , \quad (ii)\ y = e^{2x} \quad , \quad (iii)\ y = \sin x \quad , \quad (iv)\ y = \cos x$

Solution:

$(i)\ y' = 6x^5 + 4x^3 - 9x^2 \quad , \quad y'' = 30x^4 + 12x^2 - 18x$

$, \quad y''' = 120x^3 + 24x - 18 \quad , \quad y^{(4)} = 360x^2 + 24 \ .$

$(ii)\ y' = 2e^{2x} \quad , \quad y'' = 4e^{2x} \quad , \quad y''' = 8e^{2x} \quad , \quad y^{(4)} = 16e^{2x}$

$(iii)\ y' = \cos x \quad , \quad y'' = -\sin x \quad , \quad y''' = -\cos x \quad , \quad y^{(4)} = \sin x$

$(iv)\ y' = -\sin x \quad , \quad y'' = -\cos x \quad , \quad y''' = \sin x \quad , \quad y^{(4)} = \cos x$

S3.3 : L'Hopital Rule

Suppose that $f(x_0) = g(x_0) = 0$, and both $f'(x_0)$ and $g'(x_0)$ exist . Then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$ if $g'(x_0) \neq 0$.

Example 3.3.1 : Find each of the following limits by using L'Hopital rule :

1. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{4x^3 - x - 3}$

2. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x + x^2}$

3. $\lim\limits_{x \to 0} \dfrac{3x - \sin x}{x}$

4. $\lim\limits_{x \to 0} \dfrac{\sqrt{4 + x} - 2}{x}$

Solution:

1. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{4x^3 - x - 3} = \lim\limits_{x \to 1} \dfrac{3x^2}{12x^2 - 1} = \dfrac{3}{12 - 1} = \dfrac{3}{11}$

2. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x + x^2} = \lim\limits_{x \to 0} \dfrac{0 + \sin x}{1 + 2x} = \dfrac{\sin 0}{1 + 0} = \dfrac{0}{1} = 0$

3. $\lim\limits_{x \to 0} \dfrac{3x - \sin x}{x} = \lim\limits_{x \to 0} \dfrac{3 - \cos x}{1} = \dfrac{3 - \cos 0}{1} = \dfrac{3 - 1}{1} = 2$

4. $\lim\limits_{x \to 0} \dfrac{\sqrt{4 + x} - 2}{x}$ $\left[\dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{4 + x}}}{1} = \frac{\frac{1}{4}}{1} = \frac{1}{4}$$

Example 3.3.2 : Find each of the following limits by using L'Hopital rule :

1. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x \sin x}$

2. $\lim\limits_{x \to 0} \dfrac{x^4 - 5x^2}{x^2 + x - \sin x}$

Solution:

1. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x\sin x}$ $\qquad \left[\dfrac{0}{0}\right]$.

$= \lim\limits_{x\to 0} \dfrac{1-\cos x}{x\cos x+\sin x}$ $\qquad$ still $\left[\dfrac{0}{0}\right]$

$= \lim\limits_{x\to 0} \dfrac{\sin x}{-x\sin x+\cos x+\cos x} = \dfrac{0}{2} = 0$

2. $\lim\limits_{x\to 0} \dfrac{x^4-5x^2}{x^2+x-\sin x}$ $\qquad \left[\dfrac{0}{0}\right]$

$= \lim\limits_{x\to 0} \dfrac{4x^3-10x}{2x+1-\cos x}$ $\qquad$ still $\left[\dfrac{0}{0}\right]$

$= \lim\limits_{x\to 0} \dfrac{12x^2-10}{2+\sin x} = \dfrac{-10}{2} = -5$.

Exercises :

In exercises 1 – 6 , find y' and y'' (the first and second derivatives with respect to x) .

1) $y=x^3+6x-5$

2) $y=3x^4-\dfrac{6}{x^2}$

3) $y=7x^2-3\sin x$

4) $y=5\sin x\cos x$

5) $y=3\tan x+4\sec x$

6) $y=2\sin x-5\cos x$

In exercises 7 – 9 , find the first and second derivatives of the given function with respect to the given variable.

7) $w=2u^4-3u+1$

8) $y=6t^4-\dfrac{4}{t}$

9) $v=t^2-8\sin t$

In exercises 10 – 12 , find y' by applying the Product Rule

10) $y=(4+x)(x^3-2)$

11) $y=(x+2)(x^3+x-4)$

12) $y=(4+x)(x^2-\frac{3}{x})$

In exercises 13 – 17 , find y' .

13) $y=\tan x-3\sin x$

14) $y=5\sin 3x^2+\sqrt{x}$

15) $y=3\sin x-e^x$

16) $y=\frac{2\sin x}{3x}$

17) $y=\frac{2\tan x-3x}{3x+4}$

In exercises 18 – 21 , find y' , y'' , y''' , and $y^{(4)}$.

18) $y=x^5+6x^4-25x$

19) $y=3\sin x$

20) $y=\cos 2x$

21) $y=e^{3x}+\ln x$

In exercises 22 – 24 , find the limit by using L'Hopital rule .

22) $\lim_{x\to 1}\frac{x-1}{3x^3-x^2-2}$

23) $\lim_{x\to 0}\frac{\sin 5x}{x}$

24) $\lim_{x\to 0}\frac{e^x-1}{\sin x}$

S3.4 : Applications of Derivatives:

Slope and Tangent Line and Normal Line :

The slope of the curve $y = f(x)$ at any point $P(x, y)$ is $y' = f'(x)$.

The tangent line to the curve $y = f(x)$ at any point $P_0(x_0, f(x_0))$ is the line whose equation $\frac{y - f(x_0)}{x - x_0} = f'(x_0)$ which pass through the point P_0 on the curve $y = f(x)$.

The normal line to the curve $y = f(x)$ at any point $P_0(x_0, f(x_0))$ is the line whose equation $\frac{y - f(x_0)}{x - x_0} = -\frac{1}{f'(x_0)}$ which pass through the point P_0 on the curve $y = f(x)$.

Example 3.4.1 : Find the slope of the curve of the function $y = f(x) = x^3 - 2x^2 + 4$ at the point $(1, 3)$. Then find the equation of each of the tangent line and the normal line to the curve at the point $(1, 3)$.

Solution :

The slope at any point $= f'(x) = 3x^2 - 4x$

$\therefore$ The slope at the point $(1, 3) = f'(1) = 3 - 4 = -1$.

$$\frac{y - f(1)}{x - 1} = f'(1) \Rightarrow \frac{y - 3}{x - 1} = -1$$

$$y - 3 = -x + 1 \quad \Rightarrow \quad y + x - 4 = 0$$

Thus the equation of the tangent line at the point $(1, 3)$ is $y + x - 4 = 0$.

$$\frac{y - f(1)}{x - 1} = -\frac{1}{f'(1)} \Rightarrow \frac{y - 3}{x - 1} = 1$$

$$y - 3 = x - 1 \quad \Rightarrow \quad y - x - 2 = 0$$

Thus the equation of the normal line at the point $(1, 3)$ is $y - x - 2 = 0$.

Example 3.4.2 : Find the slope of the curve of the function $y = g(x) = x^2$ at the point $(3, 9)$. Then find the equation of each of the tangent line and the normal line to the curve at the point $(3, 9)$.

Solution : $g'(x) = 2x$

The slope of the curve at the point $(3,9)$ is $g'(3) = 2(3) = 6$.

$$\frac{y - g(3)}{x - 3} = g'(3) \quad \Rightarrow \quad \frac{y - 9}{x - 3} = 6 \quad \Rightarrow$$

$$y - 9 = 6x - 18 \quad \Rightarrow \quad y - 6x + 9 = 0$$

Thus the equation of the tangent line at the point $(3,9)$ is $y - 6x + 9 = 0$.

$$\frac{y - g(3)}{x - 3} = -\frac{1}{g'(3)} \quad \Rightarrow \quad \frac{y - 9}{x - 3} = -\frac{1}{6}$$

$$\Rightarrow \quad 6y - 54 = -x + 3 \quad \Rightarrow \quad 6y + x - 57 = 0$$

Thus the equation of the normal line at the point $(3,9)$ is $6y + x - 57 = 0$.

Exercise 3.4.3 : Find the slope of the curve of the function $y = h(x) = 3x^2 - 1$ at the point $(-1,2)$. Then find the equation of each of the tangent line and the normal line to the curve at the point $(-1,2)$.

Exercise 3.4.4 : Find the slope of the curve of the function $y = f(x) = x^3 - 4$ at the point $(2,4)$. Then find the equation of each of the tangent line and the normal line to the curve at the point $(2,4)$.

CH4 : Integration

S4.1 : The Indefinite Integral

Definition : A function $F(x)$ is anti-derivative of a function $f(x)$ with respect to x if $\frac{d}{dx}F(x) = f(x)$ for all x in the domain of f . The set of all anti derivatives of f is the indefinite integral of f with respect to x , denoted by $\int f(x)\,dx$ $i.e.$ $\int f(x)\,dx = F(x) + c$.

The symbol $\int$ is an integral sign .

The function f is the integrand of the integral and x is the variable of the integration .

Example 4.1.1: $\int 3x^2\,dx = x^3 + c$.

Integral Formulas :

1) $\int u^n\,du = \frac{u^{n+1}}{n+1} + c$, $n \neq -1, n$ rational

$\int du = \int 1\,du = u + c$ (special case)

2) $\int \sin u\,du = -\cos u + c$

3) $\int \cos u\,du = \sin u + c$

4) $\int \sec^2 u\,du = \tan u + c$

5) $\int \csc^2 u\,du = -\cot u + c$

6) $\int \sec u \tan u\,du = \sec u + c$

7) $\int \csc u \cot u\,du = -\csc u + c$

8) $\int \frac{1}{u}\,du = \ln |u| + c$

9) $\int e^u\,du = e^u + c$

10) $\int a^u\,du = \frac{a^u}{\ln a} + c$, $a > 0$

Rules of Indefinite Integration :

1) $\int k\, f(x)\, dx = k \int f(x)\, dx$

2) $\int -f(x)\, dx = -\int f(x)\, dx$

3) $\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

Example 4.1.2 :

1) $\int dx = x + c$

2) $\int x^5\, dx = \frac{x^6}{6} + c$

3) $\int \sin x\; dx = -\cos x + c$

4) $\int \cos x\; dx = \sin x + c$

5) $\int \sec^2 x\; dx = \tan x + c$

6) $\int \csc^2 x\; dx = -\cot x + c$

7) $\int \sec x\, \tan x\; dx = \sec x + c$

8) $\int \csc x\, \cot x\; dx = -\csc x + c$

Example 4.1.3 : Find each of the following :

1) $\int (x^3 + 7)^{14} . 3x^2\, dx$

2) $\int (x^2 + 4x + 5)^{10} (x + 2)\, dx$

3) $\int \sin(3x)\; dx$

4) $\int 2x \sin(x^2)\; dx$

5) $\int \sin^3 x\, \cos x\; dx$

6) $\int 2 \cos 2x\; dx$

7) $\int x^2 \cos(x^3)\; dx$

8) $\int \sec^2(7x)\; dx$

9) $\int \csc^2(6x)\; dx$

10) $\int \csc(5x)\, \cot(5x)\; dx$

Example 4.1.4 : Find each of the following :

1) $\int \frac{1}{x}\,dx$

2) $\int \tan x\ dx$

3) $\int \cot x\ dx$

4) $\int \frac{x+1}{x^2+3x+2}\,dx$

5) $\int e^x\,dx$

6) $\int 2x\,e^{x^2}\,dx$

7) $\int (7x^2 - 5\,e^{7x})\,dx$

Solution :

1) $\int \frac{1}{x}\,dx = \ln|x| + c$

2) $\int \tan x\ dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + c$

3) $\int \cot x\ dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$

4) $\int \frac{x+1}{x^2+3x+2}\,dx = \int \frac{x+1}{(x+2)(x+1)}\,dx = \int \frac{1}{(x+2)}\,dx = \ln|x+2| + c$

5) $\int e^x\,dx = e^x + c$

6) $\int 2x\,e^{x^2}\,dx = e^{x^2} + c$

7) $\int (7x^2 - 5\,e^{7x})dx = \int 7x^2\,dx - \int 5e^{7x}\,dx = \frac{7x^3}{3} - \frac{5e^{7x}}{7} + c$

Exercise 4.1.5 : Find each of the following :

1) $\int \cos^4 x\ \sin x\ dx$

2) $\int \sec^2(3x)\ dx$

3) $\int x^4 \sec^2(x^5)\ dx$

4) $\int \sec^2 x\ \tan x\ dx$

5) $\int \sec^2 x\ \tan^2 x\ dx$

6) $\int \sec^4 x\ \tan x\ dx$

7) $\int x^9 \csc^2(x^{10})\ dx$

S4.2 : The Definite Integral

Definition : If f is a continuous at every point of $[a,b]$ and if F is any anti-derivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

is called the definite integral .

Example 4.2.1 : Evaluate the integral $\int_1^4 (x^3 + 2x + 9)\,dx$

Solution : $\int_1^4 (x^3 + 2x + 9)\,dx = \left[\frac{x^4}{4} + x^2 + 9x\right]_1^4$

$$= \left(\frac{256}{4} + 16 + 36\right) - \left(\frac{1}{4} + 1 + 9\right)$$

$$= 116 - 10.25 = 105.75 \quad .$$

Example 4.2.2 : Evaluate $\int_0^{\frac{\pi}{2}} \sin x\,dx$

Solution : $\int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = 0 - (-1) = 1$

How to Find the Area :

To find the area between the graph of $y = f(x)$ and the $x - axis$ over the interval $[a,b]$ we should follow the following steps :

Step 1 : Partition $[a,b]$ with the zeros of f .

Step 2 : Integrate f over each subinterval .

Step 3 : Add the absolute values of the Integrals .

Example 4.2.3 : Find the total area of the region between the curve $y = x^2 + 2x$ and the $x - axis$ over the interval $[-3,4]$.

Solution : $x^2 + 2x = 0 \Rightarrow x(x+2) = 0$

$\Rightarrow x = 0 \quad or \quad x = -2$

$$\therefore the\ area = \left|\int_{-3}^{-2}(x^2 + 2x)\,dx\right| + \left|\int_{-2}^{0}(x^2 + 2x)\,dx\right| + \left|\int_0^4(x^2 + 2x)\,dx\right|$$

$$= \left|\left[\frac{x^3}{3} + x^2\right]_{-3}^{-2}\right| + \left|\left[\frac{x^3}{3} + x^2\right]_{-2}^{0}\right| + \left|\left[\frac{x^3}{3} + x^2\right]_0^4\right|$$

$$= \left|\left(-\frac{8}{3}+4\right)-\left(-\frac{27}{3}+9\right)\right|+\left|(0+0)-\left(-\frac{8}{3}+4\right)\right|$$

$$+\left|\left(\frac{64}{3}+16\right)-(0+0)\right|$$

$$= \frac{4}{3}+\frac{4}{3}+\frac{112}{3} = \frac{120}{3} = 40 \; .$$

Example 4.2.4 : Find the total area of the region between the curve $y = x^3 - 4x^2 + 3x$ and the $x - axis$ over the interval $[0,2]$.

Solution : $x^3 - 4x^2 + 3x = 0 \Rightarrow x\,(x^2 - 4x + 3) = 0$
$\Rightarrow x\,(x-1)\,(x-3) = 0 \Rightarrow x = 0\ (neglected)\ or\ x = 1$

$$or\ x = 3\ (neglected)$$

$$\therefore\ the\ area = \left|\int_0^1 (x^3 - 4x^2 + 3x)\,dx\right| + \left|\int_1^2 (x^3 - 4x^2 + 3x)\,dx\right|$$

$$= \left|\left[\frac{x^4}{4}-\frac{4x^3}{3}+\frac{3x^2}{2}\right]_0^1\right| + \left|\left[\frac{x^4}{4}-\frac{4x^3}{3}+\frac{3x^2}{2}\right]_1^2\right|$$

$$= \left|\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}\right)-0\right| + \left|\left(\frac{16}{4}-\frac{32}{3}+\frac{12}{2}\right)-\left(\frac{1}{4}-\frac{4}{3}+\frac{3}{2}\right)\right|$$

$$= \left|\frac{3-16+18}{12}\right| + \left|\frac{48-128+72}{12} - \frac{3-16+18}{12}\right|$$

$$= \left|\frac{5}{12}\right| + \left|-\frac{8}{12}-\frac{5}{12}\right| = \left|\frac{5}{12}\right| + \left|-\frac{13}{12}\right| = \frac{5}{12}+\frac{13}{12} = \frac{18}{12} = 1.5$$

How to Find the Area Between Two Curves over an Interval [a , b] :

To find the area between the two curves $f(x)$ and $g(x)$ over the interval $[a,b]$ we should follow the following steps :

Step 1 : Partition $[a,b]$ with the zeros of $f - g$.

Step 2 : Integrate $f - g$ over each subinterval .

Step 3 : Add the absolute values of the Integrals .

Example 4.2.5 : Find the total area of the region between the two curves $f(x) = x^2$ and $g(x) = 2x$ over the interval $[-1,2]$.

Solution : $f(x) - g(x) = x^2 - 2x = 0 \Rightarrow x\,(x-2) = 0$

$x = 0$ or $x = 2$ $(neglected)$

$\therefore$ *the area* $= \left|\int_{-1}^{0}(x^2-2x)\,dx\right| + \left|\int_{0}^{2}(x^2-2x)\,dx\right|$

$$= \left|\left[\frac{x^3}{3}-x^2\right]_{-1}^{0}\right| + \left|\left[\frac{x^3}{3}-x^2\right]_{0}^{2}\right|$$

$$= \left|(0-0)-\left(-\frac{1}{3}-1\right)\right| + \left|\left(\frac{8}{3}-4\right)-(0-0)\right|$$

$$= \left|+\frac{4}{3}\right| + \left|-\frac{4}{3}\right| = \frac{4}{3}+\frac{4}{3} = \frac{8}{3}$$

How to Find the Area Between Two Curves :

To find the area between the two curves $f(x)$ *and* $g(x)$ we should follow the following steps :

Step 1 : Find the zeros of $f-g$, and let them be a *and* b .

Step 2 : Integrate $f-g$ over the interval $[a,b]$.

Step 3 : Find the absolute value of the Integration found in step 2 .

Example 4.2.6 : Find the area of the region enclosed by the parabola $y=x^2-2$ *and the line* $y=x$.

Solution : $f(x)-g(x)=(x^2-2)-x=0 \Rightarrow x^2-2-x=0$

$$\Rightarrow (x-2)(x+1)=0 \Rightarrow x=2 \;\; or \;\; x=-1 .$$

$\therefore$ *the area* $= \left|\int_{-1}^{2}(x^2-2-x)\,dx\right|$

$$= \left|\left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{2}\right|$$

$$= \left|\left(\frac{8}{3}-\frac{4}{2}-4\right)-\left(-\frac{1}{3}-\frac{1}{2}+2\right)\right|$$

$$= \left|\frac{8-6-12}{3}-\frac{-2-3+12}{6}\right|$$

$$= \left|-\frac{10}{3}-\frac{7}{6}\right| = \left|\frac{-20-7}{6}\right| + \left|-\frac{27}{6}\right| = \frac{27}{6} = 4\,\frac{1}{2} .$$

Example 4.2.7 : Find the total area of the region enclosed by the parabola $f(x)=x^2$ *and the line* $g(x)=2x$.

Solution : $f(x)-g(x)=x^2-2x=0 \Rightarrow x(x-2)=0$

$$\Rightarrow x=0 \;\; or \;\; x=2$$

$\therefore$ *the area* $= \left| \int_0^2 (x^2 - 2x) \, dx \right|$

$$= \left| \left[\frac{x^3}{3} - x^2 \right]_0^2 \right|$$

$$= \left| \left(\frac{8}{3} - 4 \right) - (0 - 0) \right|$$

$$= \left| \frac{8-12}{3} \right| = \left| -\frac{4}{3} \right| = \frac{4}{3}$$

Rules for definite Integrals

1) Order of integration :

$$\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx$$

2) Zero integration :

$$\int_a^a f(x) \, dx = 0$$

3) Constant multiple :

$$\int_a^b k \, f(x) \, dx = k \int_a^b f(x) \, dx \quad \forall \;\; k \in R \;, \;\; and \;\; thus$$

$$\int_a^b - f(x) \, dx = - \int_a^b f(x) \, dx \quad for \;\; k = -1 \;.$$

4) Sum and difference :

$$\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

5) Additively :

$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$$

Example 4.2.8 : Suppose that

$\int_{-2}^{1} f(x) \, dx = 4$, $\int_1^3 f(x) \, dx = -3$ and $\int_{-2}^{1} h(x) \, dx = 6$. Find

1) $\int_3^1 f(x) \, dx$

2) $\int_{-2}^{1} (2 f(x) + 5 \, h(x)) dx$

3) $\int_{-2}^{3} f(x)\,dx$

4) $\int_{-2}^{1} (3\,f(x) - 2\,h(x))dx$

<u>**Solution :**</u>

1) $\int_{3}^{1} f(x)\,dx = -\int_{1}^{3} f(x)\,dx = -(-3) = 3$.

2) $\int_{-2}^{1} (2\,f(x) + 5\,h(x))dx = \int_{-2}^{1} 2\,f(x)\,dx + \int_{-2}^{1} 5\,h(x)\,dx$

$$= 2\int_{-2}^{1} f(x)\,dx + 5\int_{-2}^{1} h(x)\,dx$$

$$= 2\,(4) + 5\,(6) = 8 + 30 = 38 \ .$$

3) $\int_{-2}^{3} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$

$$= 4 + (-3) = 1 \ .$$

4) $\int_{-2}^{1} (3\,f(x) - 2\,h(x))dx = \int_{-2}^{1} 3\,f(x)\,dx - \int_{-2}^{1} 2\,h(x)\,dx$

$$= 3\int_{-2}^{1} f(x)\,dx - 2\int_{-2}^{1} h(x)\,dx$$

$$= 3\,(4) - 2\,(6) = 12 - 12 = 0 \ .$$

<u>**Exercise 4.2.9 :**</u> Evaluate the following integrals :

1) $\int_{-2}^{0} (2\,x + 5)\,dx$

2) $\int_{0}^{1} \left(x^2 + \sqrt{x} \right) dx$

3) $\int_{0}^{\pi} (1 + \cos x)\,dx$

4) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (8\,y^2 + \sin y)\,dy$

5) $\int_{2}^{1} \frac{2}{x^2}\,dx$

CH5 : Sequences and Series

S5.1 : Arithmetic Sequence and Geometric Sequence

Definition 5.1.1 : A sequence of numbers is a set of numbers arranged in a specific order.

Each number is called a term in the sequence. The first number is called the first term and will be denoted by a_1, the second number is called the second term and will be denoted by a_2, ..., the nth number is called the nth term and will be denoted by a_n.

The sequence will be written as $a_1, a_2, \ldots, a_n, \ldots$ and will be denoted by $\{a_n\}$.

Definition 5.1.2 : If $\{a_n\}$ is a given sequence and S_n is defined by

$$S_1 = a_1 ,$$

$$S_2 = a_1 + a_2 ,$$

$$S_3 = a_1 + a_2 + a_3 ,$$

$$\vdots$$

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

Then the sequence $\{S_n\}$ will be called an infinite series and will be written as

$$\sum_{i=1}^{\infty} a_i ,$$

and the terms S_n will be called the partial sum of the series.

Definition 5.1.3 : A sequence of numbers in which each term after the first term is obtained by adding a fixed number that is added is called the common difference and will be denoted by d.

<u>Example 5.1.4 :</u> 7, 10, 13, 16, 19, ... is an arithmetic sequence, since each term after the first term is obtained by adding 3 to the previous term.

In this example we have $d = 3$ and

$a_1 = 7$

$a_2 = a_1 + d = 7 + 3 = 10$

$a_3 = a_2 + d = 10 + 3 = 13$

$a_4 = a_3 + d = 13 + 3 = 16$

$\vdots$

<u>Example 5.1.5 :</u> Find the common difference d for the following arithmetic sequence

3, 9, 15, 21, 27, ...

<u>Solution :</u>

$d = 9 - 3 = 6$

or $d = 15 - 9 = 6$

or $d = 21 - 15 = 6$

or $d = 27 - 21 = 6$

<u>Example 5.1.6 :</u> Write the first seven terms of the arithmetic sequence whose first term $a_1 = 4$ and common difference $d = 9$.

<u>Solution :</u> The first term $a_1 = 4$

The second term $a_2 = a_1 + d = 4 + 9 = 13$

The third term $a_3 = a_2 + d = 13 + 9 = 22$

The fourth term $a_4 = a_3 + d = 22 + 9 = 31$

The fifth term $a_5 = a_4 + d = 31 + 9 = 40$

The sixth term $a_6 = a_5 + d = 40 + 9 = 49$

The seventh term $a_7 = a_6 + d = 49 + 9 = 58$.

<u>Exercise 5.1.7 :</u> Write the first nine terms of each of the following arithmetic sequences if you know that

(1) $a_1 = 3$ and $d = -2$

(2) $a_1 = 7$ and $d = 5$

(3) $a_2 = 10$ and $d = 4$

(4) $a_4 = 12$ and $a_5 = 17$.

<u>Remark 5.1.8 :</u> In the arithmetic sequence with the first term a_1 and common difference d, the nth term a_n is given by $a_n = a_1 + (n-1)d$

<u>Example 5.1.9 :</u> Find a_{10}, a_{12} and a_n for the arithmetic sequence $-2, 5, 12, \ldots$

<u>Solution :</u>

$a_1 = -2$ and $d = 5 - (-2) = 5 + 2 = 7$.

$\therefore a_{10} = -2 + (10-1) \times 7 = -2 + 9 \times 7 = -2 + 63 = 61$.

$a_{12} = -2 + (12-1) \times 7 = -2 + 11 \times 7 = -2 + 77 = 75$.

$a_n = -2 + (n-1) \times 7 = -2 + 7n - 7 = 7n - 9$.

<u>Exercise 5.1.10 :</u> For each of the following arithmetic sequences, find d, a_{15}, a_n :

(1) 2, 5, 8, 11, ...

(2) 5, 9, 13, 17, ...

(3) −5, 6, 17, 28, ...

S 5.2 : Arithmetic Series and Geometric Series

Definition 5.2.1 : If $\{a_n\}$ is an arithmetic sequence, then the corresponding series $\sum_{i=1}^{\infty} a_i$ is called an arithmetic series and the terms

$S_n = \sum_{i=1}^{n} a_i$ is called the nth partial sum of the arithmetic series.

Theorem (1) :

(i) The nth partial sum of an arithmetic series is

$$S_n = \frac{n}{2}(a_1 + a_n).$$

(ii) The nth partial sum of an arithmetic series is

$$S_n = n a_1 + \frac{n(n-1)}{2} d.$$

Example 5.2.2 : Find the sum of the first 100 positive integers.

Solution : $a_1 = 1$ and $a_{100} = 100$

$\therefore S_{100} = \frac{100}{2}(1 + 100) = 50(101) = 5050.$

Example 5.2.3 : Find the sum of the first 20 terms of the arithmetic sequence (the 20th partial sum of the arithmetic series) 10, 16, 22 ...

Solution :

$a_1 = 10$ and $d = 16 - 10 = 6$

$$\therefore S_{20} = 20 \times 10 + \frac{20(20-1)}{2} \times 6 = 200 + \frac{20(19)}{2} \times 6$$
$$= 200 + 190 \times 6$$
$$= 200 + 1140 = 1340 .$$

Example 5.2.4 : The sum of the first 16 terms of an arithmetic sequence is 80 . If $a_{16} = 20$, find a_1 and d.

Solution :

$$S_{16} = \frac{16}{2}(a_1 + a_{16}) = 8(a_1 + 20) = 8a_1 + 160$$

$$\therefore 80 = 8a_1 + 160 \Rightarrow 8a_1 = 80 - 160$$
$$\Rightarrow 8a_1 = -80$$
$$\Rightarrow a_1 = \frac{-80}{8} = -10$$

Since $a_{16} = a_1 + (16-1)d$ then $20 = -10 + 15d$

$$\Rightarrow 15d = 20 + 10 \Rightarrow 15d = 30 \Rightarrow d = \frac{30}{15} = 2 .$$

Example 5.2.5 : Find the sum of the first 12 terms of the sequence 11 , 18 , 25 , ...

Solution : $a_1 = 11$ and $d = 18 - 11 = 7$.

The ith term $a_i = a_1 + (i-1)d = 11 + (i-1) \times 7$

$$= 11 + 7i - 7 = 7i + 4$$

$$\therefore S_n = \sum_{i=1}^{n} (7i + 4)$$

$$\therefore S_{12} = \sum_{i=1}^{12} (7i + 4) = \frac{12}{2}(a_1 + a_{12}) = 6(11 + 88)$$
$$= 6 \times 99 = 594 .$$

Exercise 5.2.6 : Find each of the following sums :

(1) $\sum_{i=1}^{20} (3i - 10)$

(3) $\sum_{i=1}^{25} (i + 9)$

(2) $\sum_{i=1}^{15} (6i + 5)$

(4) $\sum_{i=1}^{13} (2i + 3)$

Definition 5.2.7 : A sequence of numbers in which each term after the first term is obtained by multiplying the previous term by a fixed nonzero real number is called a geometric sequence.
The fixed nonzero real number that is multiplied is called the common ratio and will be denoted by r.

Example 5.2.8 : 3, 6, 12, 24, 48, ... is a geometric sequence, since each term after the first term is obtained by multiplying the previous term by 2.
In this example we have $r = 2$ and

$a_1 = 3$

$a_2 = r \cdot a_1 = 2(3) = 6$

$a_3 = r \cdot a_2 = 2(6) = 12$

$a_4 = r \cdot a_3 = 2(12) = 24$

$\vdots$

Example 5.2.9: Find the common ratio r for the following geometric sequence

$$7, 21, 63, 189, \ldots$$

Solution : $r = \frac{21}{7} = 3$

or $r = \frac{63}{21} = 3$

or $r = \frac{189}{63} = 3$

Example 5.2.10 : Write the first five terms of the geometric sequence whose first term $a_1 = 3$ and common ratio $r = -2$.

Solution : The first term $a_1 = 3$

The second term $a_2 = r \cdot a_1 = (-2) \times 3 = -6$

The third term $a_3 = r \cdot a_2 = (-2)(-6) = 12$

The fourth term $a_4 = r \cdot a_3 = (-2) \times 12 = -24$

The fifth term $a_5 = r \cdot a_4 = (-2)(-24) = 48$.

Exercise 5.2.11 : Write the first seven terms of each of the following geometric sequence if you know that

(1) $a_1 = -6$ and $r = 2$

(2) $a_1 = 7$ and $r = -1$

(3) $a_1 = 4$ and $r = 5$

Remark 5.2.12 : In the geometric sequence with the first term a_1 and the common ratio r, the nth term a_n is given by $a_n = r^{n-1} \times a_1$

Example 5.2.13 : Find a_5, a_7 and a_n for the geometric sequence 5, 10, 20, ...

Solution :

$a_1 = 5$ and $r = \frac{10}{5} = 2$.

$\therefore a_5 = 2^{5-1} \times 5 = 2^4 \times 5 = 16 \times 5 = 80$

$a_7 = 2^{7-1} \times 5 = 2^6 \times 5 = 64 \times 5 = 320$

$a_n = 2^{n-1} \times 5$

Exercise 5.2.14 : For each of the following geometric sequences, find r, a_6 and a_n.

(1) 6, 18, 54, ...

(2) 7, −7, 7, ...

(3) −2, 4, −8, ...

Definition 5.2.15 : If $\{a_n\}$ is a geometric sequence, then the corresponding series $\sum_{i=1}^{\infty} a_i$ is called a geometric series and the terms

$$S_n = \sum_{i=1}^{n} a_i = a_1 + ra_1 + r^2 a_1 + \cdots + r^{n-1} a_1$$ is called

the nth partial sum of the geometric series.

Theorem (2) :

(i) The nth partial sum of a geometric series is

$$S_n = \begin{cases} \dfrac{a_1(1-r^n)}{1-r} & \text{if } r \neq 1 \\ n a_1 & \text{if } r = 1 \end{cases}$$

(ii) The nth partial sum of a geometric series is

$$S_n = \begin{cases} \dfrac{a_1 - r a_n}{1-r} & \text{if } r \neq 1 \\ n a_1 & \text{if } r = 1 \end{cases}$$

Example 5.2.16 : Find the sum S_7 of the first seven terms of the geometric sequence (the 7th partial sum of the geometric series) 4, 8, 16, ...

Solution:

$r = \frac{8}{4} = 2$, $a_1 = 4$

∴ The sum of the first seven terms is

$$S_7 = \frac{4(1-2^7)}{1-2} = \frac{4(1-128)}{-1} = \frac{4(-127)}{-1}$$

$$= \frac{-508}{-1} = 508 .$$

Example 5.2.17: Find $\sum_{i=1}^{5} 7(4)^i$.

Solution: $a_1 = 7(4)^1 = 28$

$a_2 = 7(4)^2 = 7(16) = 112$

$\therefore r = \frac{a_2}{a_1} = \frac{112}{28} = 4$.

$$\therefore \sum_{i=1}^{5} 7(4)^i = S_5 = \frac{28(1-4^5)}{1-4} = \frac{28(1-1024)}{-3}$$

$$= \frac{28(-1023)}{-3} = \frac{-28644}{-3} = 9548 .$$

Exercise 5.2.18: Find each of the following:

(1) $\sum_{i=1}^{7} 2(3)^i$

(2) $\sum_{i=1}^{6} 6(2)^i$

(3) $\sum_{i=1}^{5} 9(-2)^i$

(4) $\sum_{i=1}^{10} 2^i$

(69)

Definition 5.2.19 : Let $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n + \cdots$ be an infinite series and let $\{S_n\}$, where $S_n = a_1 + a_2 + \cdots + a_n$ for $n = 1, 2, 3, \ldots$ be the sequence of partial sums of the infinite series.

If $\lim_{n\to\infty} S_n$ exists and equals a number S, the series is said to be convergent (and to converge to the value S) and S is called the sum of the infinite series $\sum_{i=1}^{\infty} a_i$.

If $\lim_{n\to\infty} S_n$ fails to exist or not a finite number, the series is divergent and has no sum.

Example 5.2.20 : Find the sum of the infinite series $\sum_{m=1}^{\infty} \frac{1}{2^m}$.

Solution :

$$S_1 = \frac{1}{2},$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8},$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

$$\vdots$$

Then
$$\sum_{m=1}^{\infty} \frac{1}{2^m} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right)$$
$$= 1 - 0$$
$$= 1$$

Remark 5.2.21 :

The geometric series converges if $|r| < 1$ and diverges if $|r| \geq 1$.

Exercise 5.2.22:

State whether each of the following series converges or diverges, and then find the sum of the series if it converges:

(1) $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

(2) $\sum_{n=1}^{\infty} \frac{5}{3^{n-1}}$

(3) $\sum_{n=1}^{\infty} 3(2^{n-1})$

(4) $\sum_{n=1}^{\infty} 7(-1)^{n-1}$

S 5.3 : Power Series, Taylor Series and Maclurian Series

Definition 5.3.1 : A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Definition 5.3.2 : The Maclurian series for a function f is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

(i.e. $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$ about $x = 0$).

Example 5.3.3 : Find the Maclurian series for the function $f(x) = e^x$.

Solution: Since $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, ..., $f^{(n)}(x) = e^x$.

Then $f(0) = e^0 = 1$, $f'(0) = e^0 = 1$, $f''(0) = e^0 = 1, \ldots, f^{(n)}(0) = e^0 = 1$, and this implies that the Maclurian series for the function $f(x) = e^x$ is

$$e^x = f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 + 1.x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}x^k .$$

Example 5.3.4 : Find the Maclurian series for the function $f(x) = \cos x$.

Solution : Since $f(x) = \cos x$, $f'(x) = -\sin x$,

$$f''(x) = -\cos x , \quad f^{(3)}(x) = \sin x ,$$

$$\vdots \qquad \vdots$$

$$f^{(2k)}(x) = (-1)^k \cos x , \quad f^{(2k+1)}(x) = (-1)^{k+1} \sin x .$$

Then $f^{(2k)}(0) = (-1)^k \cos 0 = (-1)^k . 1 = (-1)^k$ and $f^{(2k+1)}(0) = (-1)^{k+1} \sin 0 = (-1)^{k+1} . 0 = 0$, and this implies that the Maclurian series for the function $f(x) = \cos x$ is

$$\cos x = f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 + 0 + \frac{(-1)}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 + \frac{(-1)}{6!}x^6 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Exercise 5.3.5 : Find the Maclurian series for the function $f(x) = \sin x$.

Definition 5.3.6 : The Taylor series for the function f about $x = a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Remark 5.3.7 : The Maclurian series are Taylor series with $a = 0$.

Example 5.3.8 : Find the Taylor series of $\cos x$ about $x = 2\pi$.

Solution :

Since $f(x) = \cos x$, $f'(x) = -\sin x$,

$f''(x) = -\cos x$, $f^{(3)}(x) = \sin x$,

$\vdots \qquad \vdots$

$$f^{(2k)}(x) = (-1)^k \cos x, \quad f^{(2k+1)}(x) = (-1)^{k+1} \sin x, \ldots$$

Then $f^{(2k)}(2\pi) = (-1)^k \cdot \cos(2\pi) = (-1)^k \cdot 1 = (-1)^k$ and $f^{(2k+1)}(2\pi) = (-1)^{k+1} \sin(2\pi) = (-1)^{k+1} \cdot 0 = 0$ and this implies that the Taylor series of $f(x) = \cos x$ about $x = 2\pi$ is

$$\cos x = f(x) = f(2\pi) + f'(2\pi)(x-2\pi) + \frac{f''(2\pi)}{2!}(x-2\pi)^2 + \frac{f^{(n)}(2\pi)}{n!}(x-2\pi)^n + \cdots$$

$$= \cos(2\pi) - \sin(2\pi)(x-2\pi) - \frac{\cos(2\pi)}{2!}(x-2\pi)^2 + \frac{\sin(2\pi)}{3!}(x-2\pi)^3 + \frac{\cos(2\pi)}{4!}(x-2\pi)^4 + \cdots$$

$$= 1 - 0(x-2\pi) - \frac{1}{2!}(x-2\pi)^2 + \frac{0}{3!}(x-2\pi)^3 + \frac{1}{4!}(x-2\pi)^4 + \cdots$$

$$= 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(x-2\pi)^{2k}}{(2k)!}.$$

Example 5.3.9: Find the Taylor series for the function $f(x) = \frac{1}{x}$ about $x = 1$.

Solution:

Since $f(x) = \frac{1}{x} = x^{-1}$, $f'(x) = -1 \cdot x^{-2} = \frac{-1}{x^2}$,

$$f''(x) = 2x^{-3} = \frac{2}{x^3} = \frac{2!}{x^3}, \quad f^{(3)}(x) = -6x^{-4} = \frac{-6}{x^4} = \frac{-3!}{x^4}$$

$$\vdots$$

$$f^{(k)}(x) = (-1)^k \cdot \frac{k!}{x^{k+1}}, \ldots$$

Then $f^{(k)}(1) = (-1)^k \cdot \frac{k!}{(1)^{k+1}} = (-1)^k \cdot k!$ and this implies that the Taylor series of $f(x) = \frac{1}{x}$ about $x = 1$ is

$$\frac{1}{x} = f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \cdots$$

$$= 1 + (-1)^1(1!)(x-1) + \frac{(-1)^2 \cdot 2!}{2!}(x-1)^2 + \frac{(-1)^3 \cdot 3!}{3!}(x-1)^3 + \frac{(-1)^4 \cdot 4!}{4!}(x-1)^4 + \cdots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k (x-1)^k .$$

Exercise 5.3.10: Find the Taylor series for the function $f(x) = \frac{1}{x}$ about $x = -1$.

(ans. $\frac{1}{x} = f(x) = \sum_{k=0}^{\infty} (-1) \cdot (x+1)^k$).

S 5.4 : Fourier Series

Definition 5.4.1: The fourier series of a function $f(x)$ defined on the interval $-L < x < L$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \quad \text{---}(1)$$

where $$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Remarks 5.4.2:

Suppose that f is a function defined over the symmetric interval $-L < x < L$. Assume that f is expressible as the trigonometric series given by equation (1). If m and n are positive integers. Then

(1) $\displaystyle\int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx = 0$

(2) $\displaystyle\int_{-L}^{L} \sin\frac{n\pi x}{L}\,dx = 0$

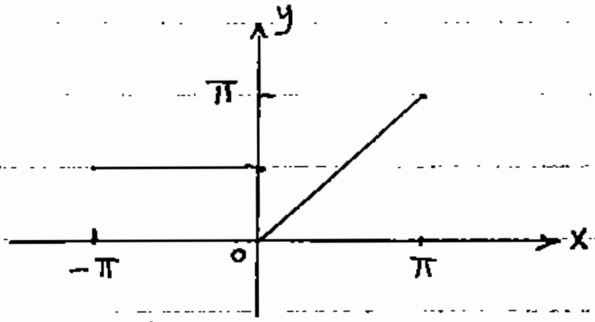
(3) $$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & m\neq n \\ L & m=n \end{cases}$$

(4) $$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0$$

(5) $$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & m\neq n \\ L & m=n \end{cases}$$

Example 5.4.3: Find the fourier series expansion of the function

$$f(x) = \begin{cases} 1 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$



Solution:

Since $L=\pi$,

then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[x\Big|_{-\pi}^{0}\right] + \frac{1}{\pi}\left[\frac{x^2}{2}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}(+\pi) + \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = 1 + \frac{\pi}{2}$$

CH6 : Hyperbolic Functions and Inverse Hyperbolic Functions

S6.1 : Hyperbolic Functions

Definition 6.1.1 :

(1) Hyperbolic cosine of x : $\cosh x = \dfrac{e^x + e^{-x}}{2}$

(2) Hyperbolic sine of x : $\sinh x = \dfrac{e^x - e^{-x}}{2}$

(3) Hyperbolic tangent: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(4) Hyperbolic cotangent : $\coth x = \dfrac{\cosh x}{\sinh x}$

$$= \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

(5) Hyperbolic secant : $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(6) Hyperbolic cosecant : $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

S 6.2 : Inverse Hyperbolic Functions

The inverse hyperbolic sine function is defined by
$y = \sinh^{-1} x$ iff $x = \sinh y$

The inverse hyperbolic cosine function is defined by
$y = \cosh^{-1} x$ iff $x = \cosh y$ and $y \geq 0$.

or by $\{(x, y) \mid x = \cosh y , y \geq 0\}$

Similarly, tanh, coth and csch have inverses, denoted by $\tanh^{-1}$, $\coth^{-1}$ and csch^{-1}

Remark 6.2.1

sech does not have a unique inverse.

We define the inverse hyperbolic secant by
$y = \operatorname{sech}^{-1} x$ iff $x = \operatorname{sech} y$ and $y \geq 0$
or $\{(x, y) \mid x = \operatorname{sech} y , y \geq 0\}$.

Remarks 6.2.2

Since the natural logarithmic function is the inverse of the exponential function, then the inverse hyperbolic functions may be expressed in terms of $\ln x$.

(80)

Let $y = \cosh^{-1} x$, where $x \geq 1$. Then

$x = \cosh y = \frac{1}{2}(e^y + e^{-y})$ for $y \geq 0$.

$\Rightarrow \quad 2x e^y = e^{2y} + 1$

$\Rightarrow \quad (e^y)^2 - 2x(e^y) + 1 = 0$

$\Rightarrow \quad e^y = x \pm \sqrt{x^2 - 1}$ or

$y = \ln(x \pm \sqrt{x^2-1})$

$\Rightarrow \quad \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ [Since $\cosh^{-1} x$ is the larger of these two values of y]

Similarly

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ (for any x),

$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ $(x \geq 1)$,

$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$ $(-1 < x < 1)$.