10: Limits and continuity

Let $f(x)$ be defined on an open interval about $x_\circ$, except possibly at $x_\circ$ itself. The limit of $f(x)$ as approaches $x_\circ$ is the number L $\lim_{x \to x_\circ} f(x) = L$.

The limit of a function $f(x)$ as $x \to x_\circ$ never depend on what happen when $x = x_\circ$

Right hand limit $\lim_{x \to x_\circ^+} f(x) = L$

Left hand limit $\lim_{x \to x_\circ^-} f(x) = L$

A function $f(x)$ has a limit at point $x_\circ$ if and only if the right and left hand limit at $x_\circ$ exit and equal

$$\lim_{x \to x_\circ^+} f(x) = L \iff \lim_{x \to x_\circ^-} f(x) = L \quad \text{and} \lim_{x \to x_\circ} f(x) = L$$

10.1 The Limit Laws:

If L, M, C and K are real number and $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$

1) Sum rule $\lim_{x \to c} (f(x) + g(x)) = L + M$
2) Deference $\lim_{x \to c} (f(x) - g(x)) = L - M$
3) Product $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$
4) Constant multiple $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$
5) Quotient rule $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6) power rule $\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$

EXAMPLE 1: Find the limit of the function $f(x) = x^3 + 4x^2 - 3$ at $x \to c$

Sol:

$$f(x) = x^3 + 4x^2 - 3$$

$$\lim_{x \to c} (x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$

$$= c^3 + 4c^2 - 3$$

H.W **Ex 2**: $\lim_{x \to c} \frac{x^4 - x^2 - 1}{x^2 + 5}$

H.W **Ex 3:** $\lim_{x \to 2} \sqrt{4x^2 - 3}$

Find the limit of function

a) The limit laws

1. $\lim_{x \to 2} (4)$
2. $\lim_{x \to 2} (5x - 3)$
3. $\lim_{x \to -2} \frac{3x + 4}{x + 5}$
4. $\lim_{x \to 1} (x^2 - 2x + 3)$ (To finding limits by calculating)
5. $\lim_{x \to 3} \sqrt{x^2 + 3x - 4}$
6. $\lim_{x \to -3} \frac{x^2 + x - 1}{2x - 4x^2}$
7. $\lim_{x \to 13} (4)$

b) Limit of Rational Function

EXAMPLE 1: $\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)}=\lim_{x\to 1}(x+1)=1+1=2$

Ex 2: $\lim_{x\to 2}\frac{x-2}{x^2-4}$

Ex 3: $\lim_{x\to -3}\frac{x^2-9}{x+3}$

Ex 4: $\lim_{x\to 2}\frac{x^4-16}{x-2}$

→ Limit of Rational function can be found by substitution if the limit of denominator is not zero

Ex 5: $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$

Ex 6: $\lim_{x\to 4}\frac{x^2-3x-4}{x-4}$

Ex 7: $\lim_{x\to -4}\frac{x^2-16}{x^2-5x+4}$

Ex 8: $\lim_{x\to 5}\frac{x^2+5-6x}{x^2-25}$

Ex 9: $\lim_{x\to 5}\frac{x^3-8}{x^2-3x+2}$

Ex 10: $\lim_{x\to 5}\frac{x^4-16}{x^3-8}$

Hint $a^3-b^3=(a-b)(a^2+ab+b^2)$

$a^3+b^3=(a+b)(a^2-ab+b^2)$

Ex 1 $(x^3-8)=(x^3-2^3)=(x-2)(x^2+2x+4)$

c) Limit at infinity of Rational function

Hint

1) $\frac{\infty}{x} = \infty$
2) $\frac{x}{\infty} = 0$
3) $\frac{\infty}{0} = \infty$
4) $\frac{0}{\infty} = \infty$

To find Limit we can divide the numerator and denominator by highest power of x in denominator

If the value of $x \to \pm\infty$

EXAMPLE 1: $\lim\limits_{x\to\infty} \frac{x^3 - 2x + 1}{2 + x^2 - 4x^3}$

Sol:

$$\lim_{x\to\infty} \frac{\frac{x^3}{x^3} - \frac{2x}{x^3} + \frac{1}{x^3}}{\frac{2}{x^3} + \frac{x^2}{x^3} - \frac{4x^3}{x^3}}$$

$$\lim_{x\to\infty} \frac{1 - \frac{2}{x^2} + \frac{1}{x^3}}{\frac{2}{x^3} + \frac{1}{x} - 4} = \frac{1 - \frac{2}{\infty} + \frac{1}{\infty}}{\frac{2}{\infty} + \frac{1}{\infty} - 4} = \frac{1 - 0 + 0}{0 + 0 - 4}$$

$$\lim_{x\to\infty} \frac{x^3 - 2x + 1}{2 + x^2 - 4x^3} = -\frac{1}{4}$$

Ex 2: $\lim\limits_{x\to\infty} \frac{4x^5 + 2x^3 + 3x - 1}{2x^7 - 3x^4 + 2x - 7}$ ans (0)

Ex 3: $\lim\limits_{x\to\infty} \frac{2x - 3}{x + \sqrt{x^2 - 1}}$ ans (1)

Ex 4: $\lim\limits_{x\to\infty} \frac{7x + 2}{3x + \sqrt{x^2 + 1}}$ ans (7/4)

d) Limit of Root function

If we have Root $\pm$ number

Root $\pm$ Root

EXAMPLE 1: Find limit of $\frac{1-\sqrt{x+1}}{x}$ at $x \to 0$

Sol:

$$\lim_{x\to 0}\frac{1-\sqrt{x+1}}{x}\cdot\frac{1+\sqrt{x+1}}{1+\sqrt{x+1}}=\lim_{x\to 0}\frac{1-(x+1)}{x+x\sqrt{x+1}}$$

$$=\lim_{x\to 0}\frac{1-x-1}{x(1+\sqrt{x+1})}=\lim_{x\to 0}\frac{-x}{x(1+\sqrt{x+1})}=\frac{-1}{(1+\sqrt{x+1})}=\frac{-1}{(1+\sqrt{0+1})}$$

$$=\frac{-1}{(1+1)}=-\frac{1}{2}$$

H.W **Ex 2** $\lim_{n\to\infty}\sqrt{n^2+1}-n$

EXAMPLE 3: $\lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

Sol:

$$\lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}=\lim_{x\to 1}\frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}\cdot\frac{\sqrt{x+1}+\sqrt{2x}}{\sqrt{x+1}+\sqrt{2x}}$$

$$\lim_{x\to 1}\frac{x+1-2x}{(x^2-x)(\sqrt{x+1}+\sqrt{2x})}=\lim_{x\to 1}\frac{1-x}{x\,(x-1)(\sqrt{x+1}+\sqrt{2x})}$$

$$=\lim_{x\to 1}\frac{-(x-1)}{x\,(x-1)(\sqrt{x+1}+\sqrt{2x})}=\lim_{x\to 1}\frac{-1}{x(\sqrt{x+1}+\sqrt{2x})}$$

$$=\lim_{x\to 1}\frac{-1}{1\cdot(\sqrt{1+1}+\sqrt{2\cdot 1})}=-\frac{1}{2\sqrt{2}}$$

H.W **Ex 4:** $\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}$

H.W **Ex 5:** $\lim_{x\to 0}\frac{x^3-a^3}{x^4-a^4}$

11. Continuity

Continuous function: A function is continuous if it is continuous at each point of its domain.

The Continuity test

The function $y = f(x)$ is continuous at $x = c$ if and only if all three of following statement are true

1. $f(c)$ exit "c in the domain of f"
2. $\lim_{x \to c} f(x)$ exit "f has a limit at $x \to c$"
3. $\lim_{x \to c} f(x) = f(c)$ "The limit equal the function value"

Hint if f continues at $x \to c$ and g continuous at $x \to c$

1. $f \cdot g$
2. $g \cdot f$
3. $k \cdot g$
4. f / g

.... continuous

EXAMPLE 1: Determine if the following function is continuous at $x = 1$?

$$f(x) = \begin{cases} 3x - 5 & \text{at } x \neq 1 \\ 2 & \text{at } x = 1 \end{cases}$$

Sol:

1) $f(1) = 2$
2) $\lim_{x \to 1} f(x) = \lim_{x \to 1} (3x - 5)$

$= (3(1) - 5) = -2$

$f(1) \neq \lim_{x \to 1} f(x)$

The function $f(x)$ is not continuous at $x = 1$

H.W **Ex 2:** Determine if the $f(x)$ is continuous at $x = 3$?

$$f(x) = \begin{cases} x^2 - 1 & \text{at } x = 3 \\ x^2 + 2 & \text{at } x \neq 3 \end{cases}$$

EXAMPLE 3: if $f(x)=\begin{cases}\dfrac{x^2-1}{x^2-2} & \text{at } x>2\\ x+2 & \text{at } x\le 2\end{cases}$ continuous at $x=2$?

Sol:

1) $f(x)=x+2 \quad \text{at} \quad x=2$

   $f(2)=2+2=4$

2) $\lim_{x\to 2} f(x)=\lim_{x\to 2}\dfrac{x^2-4}{x-2}=\lim_{x\to 2}\dfrac{(x-2)(x+2)}{x-2}$

   $\lim_{x\to 2}(x+2)=2+2=4$

   $f(2)=\lim_{x\to 2}(x+2)$

   $f(x)$ is continuous

H.W **Ex 4:** if $f(x)=\dfrac{x+3}{x^2-1}$ where is $f(x)$ continuous, and where it is discontinuous ?

EXAMPLE 5: if $f(x)=\begin{cases}x^2+2ax+3 & x=2\\ x+4 & x\neq 2\end{cases}$ find the value a if $f(x)$ continuous.

Sol:

1) $f(x)=x^2+2ax+3$

   $f(2)=2^2+2a(2)+3$

   $f(2)=7+4a$

2) $\lim_{x\to 2}(x+4)=(2+4)=6$

If $f(x)$ is continuous

$f(2)=\lim_{x\to 2}(x+4)$

$7+4a=6$

$4a=-1 \quad \Rightarrow \quad a=-1/4$

EXAMPLE 6: For what values of x if the function $f(x)=\dfrac{x^2+3x+5}{x^2+3x-4}$ continuous?

Sol:

$x^2+3x-4=0$

$(x+4)(x-1)=0$

$x+4=0 \quad \Rightarrow \quad x=-4$

$x-1=0 \quad \Rightarrow \quad x=1$

$f(x)$ is continuous for all values of x except $x=-4$, $x=1$

EXAMPLE 7: Discuss the Continuity of

$$f(x)=\begin{cases} x+1/x & x<0 \\ -x^3 & 0\le x<1 \\ -1 & 1\le x<2 \\ 1 & x=2 \\ 0 & x>2 \end{cases}$$

Sol:

1. $x=0$

2. $x\to 0^-$, $x\to 0^+$ $\quad x\neq 0$

Case (1)

1) $f(x)=-x^3$

$f(0)=(-0)^3=0$

2) $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}(x+1/x)=(0+1/0)=\infty$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}(-x^3)=0$

$\lim\limits_{x\to 0^-} f(x)\neq \lim\limits_{x\to 0^+} \quad \Rightarrow f(x)$ discontinuous

1. $x=1$

2. $x\to 1^-$, $x\to 1^+$ $\quad x\neq 1$

Case (2)

1) $f(x)=f(1)=-1$

2) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-x^3) = -1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (-1) = -1$

The function is continuous $x=1$

1. $x=2$

2. $x \to 2^-$ $\quad x \neq 2$
   $x \to 2^+$

Case (3)

1) $f(x) = f(2) = 1$

2) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (-1) = -1$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (0) = 0$

$\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$

$f(x)$ is discontinuous at $x=2$

EXAMPLE 8: Determine all values of the constant a so that the following function is continuous for all values of x $\quad f(x) = \begin{cases} a^2x - a & x \geq 3 \\ 4 & x < 3 \end{cases}$

Sol:

$f(x) = a^2x - a$ is continuous for $x \geq 3$ at any value of a also $f(x) = 4$ is continuous for $x < 3$ that mean $f(x)$ must be defined at $x = 3$

1) $f(3) = (3a^2 - a)$

2) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (4) = 4$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$

$3a^2 - a = 4$

$3a^2 - a - 4 = 0$

$(a+1)\ (3a-4) = 0$

$a+1=0 \ \Rightarrow\ a=-1$

$3a-4=0 \ \Rightarrow a = 4/3$

H.W **Ex 9:** for what values of x is the following function continuous ?

$$f(x)=\begin{cases}\dfrac{x-1}{\sqrt{x}-1} & \text{if } x>1\\ 5-3x & -2\le x\le 1\\ \dfrac{6}{x-4} & x<-2\end{cases}$$

H.W **Ex 10:** Let $f(x)=\begin{cases}x^2+2 & x<0\\ ax+b & 0\le x<1\\ 3+2x-x^2 & x\ge 1\end{cases}$

Determine a and b so that the function $f(x)$ is continuous everywhere.

Sol:

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(x^2+2)=0^2+2=2 \qquad x<0$$

$f(x)$ must be $=2$ $\rightarrow$ $x=0$

$f(x)=ax+b$

$ax+b=2$

$a(0)+b=2 \quad \Rightarrow \quad b=2$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(3+2x-x^2)$$

$=(3+2(1)-1^2)$ at $x=1$

$=4$

$f(x)$ must be $=4$ $\rightarrow$ at $x=1$

$ax+b=4$

$a(1)+2=4$

$a=2$

H.W **Ex 11:** Determine if the following function is continuous at $x=0$

$$f(x)=\begin{cases} \dfrac{x-6}{x-3} & \text{if } x<0 \\ 2 & \text{if } x=0 \\ \sqrt{4+x^2} & \text{if } x>0 \end{cases}$$

H.W **Ex 12:** Determine if the $f(x)=\begin{cases} x^2-4x+1 & \text{at } x\geq 1 \\ 2x-3 & \text{at } x<1 \end{cases}$ continuous at $x=1$

H.W Ex 13: Suppose that $f(x)=x^3-3x^2-4x+12$ and

$$h(x)=\begin{cases} \dfrac{f(x)}{x-3} & \text{for } x\neq 3 \\ k & \text{for } x=3 \end{cases}$$

Find: a) all zeros of f

b) the value of k that h continuous at $x=3$