limit of trigonometric function

1. $\lim_{x\to 0} \sin x = 0$ $\qquad \sin(0) = 0$
2. $\lim_{x\to 0} \cos x = 1$ $\qquad \cos(0) = 1$
3. $\lim_{x\to 0} \tan x = 0$ $\qquad \tan(0) = 0$
4. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
5. $\lim_{x\to 0} \frac{\sin ax}{ax} = \lim_{x\to 0} \frac{ax}{\sin ax} = 1$
6. $\lim_{x\to 0} \frac{\tan ax}{ax} = \lim_{x\to 0} \frac{ax}{\tan ax} = 1$
7. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

EXAMPLE 1: Prove $\lim_{x\to 0} \sin x = 0$

Sol: $\lim_{x\to 0} \sin x = \sin(0) = 0$

EXAMPLE 2: Prove $\lim_{x\to 0} \cos x = 1$

Sol: $\lim_{x\to 0} \cos x = \cos(0) = 1$

EXAMPLE 3: Prove $\lim_{x\to 0} \tan x = 0$

Sol: $\lim_{x\to 0} \tan x = \tan(0) = 0$

EXAMPLE 4: Prove $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Sol: the Taylor series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$$

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\left(\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots}{x}\right)$$

at $x\to 0$ $\sin x\approx x$ so we can neglected $\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots$

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{x}{x}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

EXAMPLE 5: Prove $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

Sol: the Taylor series for $\cos x$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\cdots$$

$$=\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\left(\frac{1-(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\cdots)}{x}\right)$$

$$=\lim_{x\to 0}\frac{1-1+\frac{x^2}{2}}{x}$$

$$=\lim_{x\to 0}\frac{\frac{x^2}{2}}{x}$$

$$=\lim_{x\to 0}\frac{x}{2}$$

$$=\lim_{x\to 0}\frac{0}{2}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0$$

EXAMPLE 6: Find the limit the following function at $x\to 0$

1) $\frac{\sin 5x}{x}$ 2) $\frac{\sin x}{7x}$ 3) $\frac{\tan 3x}{x}$ 4) $\frac{\sin 5x}{7x}$ 5) $\frac{2x}{4\tan(7x)}$

6) $\frac{\sin 2x}{2x^2+x}$ 7) $\frac{\sin 3x - \tan 2x}{5x}$ 8) $\frac{\sin 7x}{\tan 3x}$ 9) $\frac{\tan^2 x}{x\sin 2x}$

10) $\sin(\pi/2\cos(\tan x))$

1) Sol:

$$= \lim_{x\to 0}\frac{\sin 5x}{x} = \lim_{x\to 0}\frac{\sin 5x}{x}\frac{5}{5}$$

$$= 5\lim_{x\to 0}\frac{\sin 5x}{5x}$$

$$= 5\,(1) = 5$$

3) Sol:

$$= \lim_{x\to 0}\frac{\tan 3x}{x} = \lim_{x\to 0}\frac{\tan 3x}{x}\frac{3}{3}$$

$$= 3\lim_{x\to 0}\frac{\tan 3x}{3x}$$

$$= 3\,(1) = 3$$

5) Sol:

$$= \lim_{x\to 0}\frac{2x}{4\tan(7x)} = \frac{7}{7}\lim_{x\to 0}\frac{2x}{4\tan(7x)}$$

$$= \frac{2}{4}\frac{1}{7}\lim_{x\to 0}\frac{7x}{\tan(7x)}$$

$$= \frac{2}{28}(1)$$

$$= \frac{1}{14}$$

6) Sol:

$$= \lim_{x\to 0}\frac{\sin 2x}{2x^2+x} = \lim_{x\to 0}\frac{\sin 2x}{x(2x+2)} = \lim_{x\to 0}\frac{\sin 2x}{x}\cdot\lim_{x\to 0}\frac{1}{2x+1}$$

$$= \lim_{x\to 0}\frac{\sin 2x}{x}\cdot\lim_{x\to 0}\frac{1}{2x+1}$$

$$= \frac{2}{2} \lim_{x\to 0} \frac{\sin 2x}{x} \cdot \lim_{x\to 0} \frac{1}{2x+1}$$
$$= 2 \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \lim_{x\to 0} \frac{1}{2x+1}$$
$$= 2(1) \cdot (\frac{1}{0+1})$$
$$= 2$$

7) Sol:

$$= \lim_{x\to 0} \frac{\sin 3x - \tan 2x}{5x} = \lim_{x\to 0} \frac{\sin 3x}{5x} - \lim_{x\to 0} \frac{\tan 2x}{5x}$$
$$= \frac{3}{3} \lim_{x\to 0} \frac{\sin 3x}{5x} - \frac{2}{2} \lim_{x\to 0} \frac{\tan 2x}{5x}$$
$$= \frac{3}{5} \lim_{x\to 0} \frac{\sin 3x}{3x} - \frac{2}{5} \lim_{x\to 0} \frac{\tan 2x}{2x}$$
$$= \frac{3}{5}(1) - \frac{2}{5}(1) = \frac{1}{5}$$

9) Sol:

$$= \lim_{x\to 0} \frac{\tan^2 x}{x \sin 2x} = \lim_{x\to 0} \frac{\left(\frac{\tan x}{x}\right)^2}{x \frac{2}{2} \frac{\sin 2x}{x^2}} = \lim_{x\to 0} \frac{\left(\frac{\tan x}{x}\right)^2}{x \frac{2 \sin 2x}{2x^2}}$$
$$= \lim_{x\to 0} \frac{\left(\frac{\tan x}{x}\right)^2}{2 \cdot \frac{\sin 2x}{2x}} = \frac{1}{2}$$

Exercises

Q1) Find the limits

a) $= \lim_{x\to -1} \frac{3x^2}{2x-1}$

b) $= \lim_{x\to \pi/2} x \sin x$

c) $= \lim_{x\to \pi} \frac{\cos x}{1-\pi}$

Q2) Calculate limits using the limit laws

a) $= \lim_{t \to 1} \frac{t^2 + 3t + 2}{t^2 - t - 2}$

b) $= \lim_{x \to -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

c) $= \lim_{u \to 1} \frac{u^4 - 1}{u^3 - 1}$

d) $= \lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$

e) $= \lim_{x \to 4} \frac{4x - x^2}{2 - \sqrt{x}}$

Q3) Using $= \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Show that a) $= \lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

b) $= \lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$