Rule of Derivatives: Let c and n are constant, u, v and ware differentiable function of x:

1. $\frac{d}{dx}c = 0$
2. $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
3. $\frac{d}{dx}(\frac{1}{u}) = -\frac{1}{u^2}\frac{du}{dx}$
4. $\frac{d}{dx}cu = c\frac{du}{dx}$
5. $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$ and $\frac{d}{dx}(u \cdot v \cdot w) = u \cdot v\frac{dw}{dx} + u \cdot w\frac{dv}{dx} + v \cdot w\frac{du}{dx}$
6. $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where $v \neq 0$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y = (x^2 + 1)^5$

Sol: $y = (x^2 + 1)^5$

$$y' = 5(x^2 + 1)^4(2x)$$

$$y' = 10x(x^2 + 1)^4$$

2. $y = \frac{x^2 - 1}{x^2 + x - 2}$

Sol:

$$y' = \frac{2x(x^2 + x - 2) - (2x + 1)(x^2 - 1)}{(x^2 + x - 2)^2}$$

$$y' = \frac{2x^3 + 2x^2 - 4x - 2x^3 + 2x - x^2 + 1}{(x^2 + x - 2)^2} = \frac{x^2 - 2x + 1}{(x^2 + x - 2)^2}$$

H.W **Ex 3:** $y = \frac{12}{x} - \frac{4}{x^2} + \frac{3}{x^4}$

H.W **Ex 4**: $y = (2x^3 - 3x^2 + 6x)^{-5}$

H.W **Ex 5**: $y = \frac{x^2 - 1}{x^2 + x - 2}$

EXAMPLE 6: $y = \frac{x^2 - 1}{x + 1}$

Sol:

$$y' = \frac{(x+1)(2x) - (x^2 - 1)}{(x+1)^2} = \frac{2x^2 + 2x - x^2 + 1}{(x+1)^2} = \frac{x^2 + 2x + 1}{(x+1)^2}$$

EXAMPLE 7: $y = \sqrt[3]{x^2} \quad \Rightarrow \quad y = x^{2/3}$

$$y' = \frac{2}{3} x^{-1/3}$$

<u>The chain rule</u>

1. Suppose that $h = g \cdot f$ is the composite of the differentiable functions $y = g(t)$ and $x = f(t)$, then h is a differentiable function of x whose derivative at each value of x is

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}}$$

EXAMPLE 1: Find $\frac{dy}{dx}$ if $y = \frac{1}{t^2 + 1}$, $x = \sqrt{4t + 1}$

Sol:

$$y = (t^2 + 1)^{-1}, \ x = \sqrt{4t + 1}$$

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{\frac{d}{dt}(t^2 + 1)^{-1}}{\frac{d}{dt}(4t + 1)^{1/2}}$$

$$= \frac{-(t^2+1)^{-2}(2t)}{\frac{1}{2}(4t+1)^{-1/2}\cdot 4} = \frac{-2t\,(t^2+1)^{-2}}{2(4t+1)^{-1/2}}$$

$$= \frac{-t\,(t^2+1)^{-2}}{(4t+1)^{-1/2}}$$

2. If y is a differentiable function of t and t is a differentiable function of x ,then y is a differentiable of x :

$y = g(t)$ and $t = f(x)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}}$$

EXAMPLE 1: Use the chain rule to express $\frac{dy}{dx}$ in terms of x and y

$$y = \frac{t^2}{t^2+1} \qquad , \qquad t = \sqrt{2x+1} = (2x+1)^{1/2}$$

Sol:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{(t^2+1)2t - t^2(2t)}{(t^2+1)^2}\cdot\frac{1}{2}(2x+1)^{-1/2}(2)$$

$$= \frac{2t^3+2t-2t^3}{(t^2+1)^2}\cdot(2x+1)^{-1/2}$$

$$= \frac{2t}{(t^2+1)^2}\cdot\frac{1}{\sqrt{2x+1}} \qquad \text{sub } t$$

$$= \frac{2\sqrt{2x+1}}{(2x+1+1)^2}\cdot\frac{1}{\sqrt{2x+1}} = \frac{2}{(2x+2)^2}$$

EXAMPLE 2: Use the chain rule to express $\frac{dy}{dx}$ in terms of x and y

$$y = \left(\frac{t-1}{t+1}\right)^2, \; x = \frac{1}{t^2} - 1 \quad \text{at} \quad t = 2$$

Sol:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \Rightarrow \qquad y = \left(\frac{t-1}{t+1}\right)^2$$

$$\frac{dy}{dt} = 2\left(\frac{t-1}{t+1}\right) \frac{t+1-(t-1)}{(t+1)^2}$$

$$\frac{dy}{dt} = \frac{4(t-1)}{(t+1)^3}$$

$$\left.\frac{dy}{dt}\right|_{t=2} = \frac{4(2-1)}{(2+1)^3} = 4/27$$

$$x = \frac{1}{t^2} - 1$$

$$\left.\frac{dx}{dt}\right|_{t=2} = \frac{-2}{t^3} = -1/4$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = 4/27 \div (-1/4) = -16/27$$

Higher derivative

If a function $y = f(x)$ possesses a derivative at every point of some interval. We may form the function $f'(x)$ and take about its derivate if it has one.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx} f'(x)$$

This derivative is called the second derivative of y with respect to x. In some manner we may define third and higher derivatives using similar notations.

EXAMPLE 1: Find all derivatives of the following function.

$$y = 3x^3 - 4x^2 + 7x + 10$$

Sol:

$$y' = 9x^2 - 8x + 7$$

$$y'' = 18x - 8$$

$$y''' = 18$$

$$y'''' = 0$$

EXAMPLE 2:

$$y = \frac{1}{x} + \sqrt{x^3} \qquad \Rightarrow y = x^{-1} + x^{3/2}$$

Sol:

$$y' = -\frac{1}{x^2} + \frac{3}{2}x^{1/2}$$

$$y'' = \frac{2}{x^3} + \frac{3}{4}x^{-1/2}$$

$$y''' = -\frac{6}{x^4} - \frac{3}{8}x^{-3/2}$$

$$= -\frac{6}{x^4} - \frac{3}{8\sqrt{x^3}}$$

Implicit derivative

If the formula of f is an algebraic combination of power of x and y.To calculate the derivative of the implicitly defined functions. We simply differentiable both sides of the defining equation with respect to x.

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $x^2y^2 = x^2 + y^2$

Sol:

$$x^2 2yy' + 2xy^2 = 2x + 2y\,y'$$
$$x^2 2yy' - 2y\,y' = 2x - 2xy^2$$
$$y'(2x^2y - 2y) = 2x - 2xy^2$$
$$y' = \frac{2x - 2xy^2}{2x^2y - 2y} = \frac{x - xy^2}{x^2y - y}$$

2. $\frac{x-y}{x-2y} = 2$

Sol:

$$2x - 4y = x - y$$
$$2 - 4y' = 1 - y'$$
$$2 - 1 = -y' + 4y'$$
$$1 = 3y'$$
$$y' = 1/3$$

3. $xy + 2x - 5y = 2$ at $(3,2)$

Sol:

$$xy' + y + 2 - 5y' = 0$$
$$y'(x-5) = -y - 2$$

$$y' = \frac{-(y+2)}{(x-5)} = \frac{-(2+2)}{(3-5)} = \frac{-4}{-2} = 2$$

EXAMPLE 2: write an equation for the tangent line at $x=3$ of the curve $f(x) = \frac{1}{\sqrt{2x+3}}$

Sol:

Equation of tangent line $y - y_1 = m\,(x - x_1)$

$y = \frac{1}{\sqrt{2x+3}}$ at $x=3$ $\Rightarrow y = \frac{1}{\sqrt{9}} = \frac{1}{3}$

P $(3,1/3)$

The slope of y' $\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(2x+3)^{-1/2}$

$y' = -1/2(2x+3)^{-3/2}\ (2)$

$y' = \frac{-1}{(2x+3)^{3/2}}$ y' at $x=3$ $= \frac{-1}{(2(3)+3)^{3/2}} = -\frac{1}{9^{3/2}}$

$y' = -\frac{1}{3^{2^{3/2}}} = -\frac{1}{3^3} = -\frac{1}{27}$

$y' = -\frac{1}{27}$

$(y - 1/3) = (-1/27)\ (x-3)$ Equation of tangent

$y - 1/3 = -1/27x + 1/9$

$y = -1/27x + 1/9 + 1/3$

$y = -1/27x + 4/9$

$y + 1/27x - 4/9 = 0$

Trigonometric function

1. $\sin u$ $\qquad \frac{d}{x}\sin u = \cos u \frac{du}{dx}$
2. $\cos u$ $\qquad \frac{d}{x}\cos u = -\sin u \frac{du}{dx}$
3. $\tan u$ $\qquad \frac{d}{x}\tan u = \sec^2 u \frac{du}{dx}$
4. $\cot u$ $\qquad \frac{d}{x}\cot u = -\csc^2 u \frac{du}{dx}$
5. $\sec u$ $\qquad \frac{d}{x}\sec u = \sec u \tan u \frac{du}{dx}$
6. $\csc u$ $\qquad \frac{d}{x}\csc u = -\csc u \cot u \frac{du}{dx}$

EXAMPLE 1: Prove that $\frac{d}{x}\tan u = \sec^2 u \frac{du}{dx}$

Sol:

$$\frac{d}{x}\tan u = \frac{d}{x}\frac{\sin u}{\cos u}$$

$$= \frac{\cos u \cos u - \sin u(-\sin u)}{\cos^2 u}\frac{du}{dx}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\frac{du}{dx}$$

$$= \frac{1}{\cos^2 u}\frac{du}{dx} = \sec^2 u \frac{du}{dx}$$

EXAMPLE 2: Prove that $\frac{d}{x}\sec u = \sec u \tan u \frac{du}{dx}$

Sol:

$$\frac{d}{x}\sec u = \frac{d}{dx}\frac{1}{\cos u} = -\frac{1}{\cos^2 u}(-\sin u)\frac{du}{dx}$$

$$= \frac{\sin u}{\cos u}\frac{1}{\cos u}\frac{du}{dx}$$

$$= \sec u \tan u \frac{du}{dx}$$

EXAMPLE 3: Find $\frac{dy}{dx}$ for the following function

1. $y = \tan(3x^2)$

Sol:

$$\frac{dy}{dx} = \sec^2(3x^2)(6x) = 6x\sec^2(3x^2)$$

2. $y = (\csc x + \cot x)^2$

Sol:

$$\frac{dy}{dx} = 2(\csc x + \cot x)(-\csc x \cot x - \csc^2 x)$$

$$= -2\csc x(\csc x + \cot x)^2$$

Hint:

1. $y = \sin^n u$ $\qquad y' = n\sin^{n-1} u \cos u \frac{du}{dx}$
2. $y = \cos^n u$ $\qquad y' = n\cos^{n-1} u(-\sin u)\frac{du}{dx}$
3. $y = \tan^n u$ $\qquad y' = n\tan^{n-1} u \sec^2 u \frac{du}{dx}$
4. $y = \cot^n u$ $\qquad y' = n\cot^{n-1} u(-\csc^2 u)\frac{du}{dx}$
5. $y = \sec^n u$ $\qquad y' = n\sec^{n-1} u(\sec u \tan u)\frac{du}{dx}$
6. $y = \csc^n u$ $\qquad y' = n\csc^{n-1} u(-\csc u \cot u)\frac{du}{dx}$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function.

1. $y = \tan^2(\cos x)$

$$y' = 2\tan(\cos x)\sec^2(\cos x)(-\sin x)$$

$$y' = -2\sin x \tan(\cos x)\sec^2(\cos x)$$

$$y = \sec^4 x - \tan^4 x$$

$$y' = 4\sec^3 x \sec x \tan x - 4\tan^3 x \sec^2 x$$

$$= 4\sec^4 x \tan x - 4\tan^3 x \sec^2 x$$

$$= \sec^2 x (4\sec^2 x \tan x - 4\tan^3 x)$$

2. $= 2\tan(x/2) - x$

$$y' = 2\sec^2(x/2)\cdot(1/2) - 1$$

$$= \sec^2(x/2) - 1$$

$$y' = \tan^2(x/2)$$

3. $y = \cot^3 x$

$$y' = 3\cot^2 x(-\csc^2 x)\cdot 1$$

$$y' = -3\cot^2 x \csc^2 x$$

4. $x + \tan(xy) = 0$

$$1 + \sec^2(xy)(xy' + y)$$

$$\sec^2(xy)xy' + \sec^2(xy)y = -1$$

$$xy' \sec^2(xy) = -(1 + y\sec^2(xy))$$

$$y' = \frac{-(1 + y\sec^2(xy))}{x\sec^2(xy)}$$

5. $y = 2\sin\frac{x}{2} - x\cos\frac{x}{2}$

$$y' = 2\cos\frac{x}{2}\cdot\frac{1}{2} - (x(-\sin\frac{x}{2})\cdot\frac{1}{2} + \cos\frac{x}{2})$$

$$= \cos\frac{x}{2} + \frac{x}{2}\sin\frac{x}{2} - \cos\frac{x}{2} = \frac{x}{2}\sin\frac{x}{2}$$

Transcendental function derivative

1- Logarithm function الدالة الوغارتيمية

(1) If $y = \ln x$ $\Rightarrow \boxed{\dfrac{d}{dx}\ln u = \dfrac{1}{u}\dfrac{du}{dx}}$

EXAMPLE: $y = \ln x$

$$y' = \frac{1}{x}$$

(2) $\boxed{\dfrac{d}{dx}\log_a u = \dfrac{d}{dx}\left(\dfrac{\ln u}{\ln a}\right) = \dfrac{1}{\ln a}\dfrac{1}{u}\dfrac{du}{dx}}$

EXAMPLE: $y = \log_a x = \dfrac{\ln x}{\ln a}$ $\Rightarrow y' = \dfrac{1}{\ln a}\dfrac{1}{x}$

(3) If $y = \log x = \dfrac{\ln x}{\ln 10}$ $\Rightarrow y' = \dfrac{1}{\ln 10}\dfrac{1}{x}$

$$\boxed{\frac{d}{dx}\log u = \frac{1}{u}\frac{1}{\ln 10}\frac{du}{dx}}$$

EXAMPLE 1: $y = \ln(\sin x - \sec x)$

$$y' = \frac{\cos x - \sec x \tan x}{\sin x - \sec x}$$

EXAMPLE 2: Find $\dfrac{dy}{dx}$ for the following function:

1. $y = \log_{10} e^x$

Sol:

$$y = x\log_{10} e \quad \Rightarrow \quad y' = \log_{10} e$$

2. $y = \log_5 (x+1)^2$

Sol:

$$y = 2\log_5 (x+1) \quad \Rightarrow \quad y = 2\frac{\ln(x+1)}{\ln 5}$$

$$y' = 2\frac{1}{(x+1)\ln 5}$$

3. $y = \log_2 (3x^2+1)^3$

Sol:

$$y = 3\log_2 (3x^2+1) \quad \Rightarrow \quad y = 3\frac{\ln(3x^2+1)}{\ln 2}$$

$$y' = 3\frac{6x}{(3x^2+1)\ \ln 2} = \frac{18x}{(3x^2+1)\ \ln 2}$$

4. $y + \ln x + \ln y = 1$ find y'

Sol:

$$y' + \frac{1}{x} + \frac{1}{y}y' = 0$$

$$y'(1+\frac{1}{y}) = -\frac{1}{x}$$

$$y'(\frac{y+1}{y}) = -\frac{1}{x}$$

$$y' = -\frac{y}{x(y+1)}$$

5. $\sin(\ln y) = \ln(x^2 - 3x + 1)$

Sol:

$$\cos(\ln y)\cdot\frac{1}{y}y' = \frac{2x-3}{x^2-3x+1}$$

$$\cos(\ln y)\,y' = \frac{y(2x-3)}{x^2-3x+1}$$

$$y' = \frac{y(2x-3)}{\cos(\ln y)(x^2-3x+1)}$$

EXAMPLE 6: If $y = \ln(t) \qquad t = \ln(x^2-1)$

Sol:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dt}\ln(t)\cdot\frac{d}{dx}\ln(x^2-1)$$

$$y' = \frac{1}{t}\cdot\frac{2x}{(x^2-1)}$$

$$t = \ln(x^2-1)$$

$$y' = \frac{1}{\ln(x^2-1)}\cdot\frac{2x}{(x^2-1)}$$

ملاحظة يمكن استخدام الدالة الوغارتيمية في ايجاد مشتقة الدوال المعقدة او الدوال الاسية

EXAMPLE 7: Find $\frac{dy}{dx}$ for the following function:

1. $y = x^x$

Sol:

$$\ln y = \ln x^x$$

$$\ln y = x\ln x$$

$$\frac{1}{y}y' = x\frac{1}{x} + \ln x \quad \Rightarrow y' = y(1+\ln x)$$

2. $y = x^{\tan x}$

Sol:

$$\ln y = \tan x \ln x$$

$$\frac{1}{y}y' = \tan x\,\frac{1}{x} + \sec^2 x \ln x$$

$$y' = y\ \left(\frac{\tan x}{x} + \sec^2 x \ln x\right)$$

3. $y = \sin x \tan x \cos x \sec x \cot x$

Sol:

$$\ln y = \ln(\sin x \tan x \cos x \sec x \cot x)$$

$$\ln y = \ln \sin x + \ln \tan x + \ln \cos x + \ln \sec x + \ln \cot x$$

$$\frac{1}{y}y' = \frac{\cos x}{\sin x} + \frac{\sec^2 x}{\tan x} + \frac{-\sin x}{\cos x} + \frac{\sec x \tan x}{\sec x} + \frac{-\csc^2 x}{\cot x}$$

$$y' = y(\cot x + \frac{\sec^2 x}{\tan x} - \tan x + \tan x - \frac{\csc^2 x}{\cot x})$$

$$y' = y(\cot x + \sec^2 x \cot x - \csc^2 x \tan x)$$

4. $y = \sqrt[3]{\frac{x \sin x}{(x-1)(x^2+1)}} \quad \Rightarrow \quad y = \left(\frac{x \sin x}{(x-1)(x^2+1)}\right)^{1/3}$

Sol:

$$\ln y = \frac{1}{3}\ln\frac{x\sin x}{(x-1)(x^2+1)}$$

$$\ln y = \frac{1}{3}[\ln x + \ln \sin x - (\ln(x-1) + \ln(x^2+1))]$$

$$\frac{1}{y}y' = \frac{1}{3}[\frac{1}{x} + \frac{\cos x}{\sin x} - \frac{1}{(x-1)} - \frac{2x}{(x^2+1)}]$$

$$y' = \frac{y}{3}(\frac{1}{x} + \cot x - \frac{1}{(x-1)} - \frac{2x}{(x^2+1)})$$

2- **Exponential function** If is u any differentiable function of x then:

1) $\frac{d}{dx} = a^u = a^u \ln a \frac{du}{dx}$

2) $\frac{d}{dx} = e^u = e^u \frac{du}{dx}$

EXAMPLE 7: Find $\frac{dy}{dx}$ for the following function:

1. $y = 2^{3x}$

   $y' = 2^{3x} 3\ln 2$

2. $y = (2^x)^2 \quad \Rightarrow \quad y = 2^{2x}$

   $y' = 2^{2x} \ln 2(2) = 2^{2x+1} \ln 2$

3. $y = x2^{x^2}$

   $y' = x2^{x^2} \ln 2(2x) + 2^{x^2} = 2^{x^2}(2x^2 \ln 2 + 1)$

4. $y = e^{\sqrt{1-5x^2}} = e^{(1-5x^2)^{1/2}}$

   $y' = e^{(1-5x^2)^{1/2}} \cdot \frac{1}{2}(1-5x^2)^{-1/2}(10x) = e^{(1-5x^2)^{1/2}} \frac{5x}{\sqrt{1-5x^2}}$

5. $y = e^{7x}$

   $y' = 7e^{7x}$

6. $y = e^{\tan x}$

   $y' = e^{\tan x} \sec^2 x$

7. $y = 3^{\tan x}$

$$y' = \ln 3\, 3^{\tan x} \sec^2 x$$

8. $y = x 2^{x^2}$

$$y' = x \ln 2\, 2^{x^2}(2x) + 2^{x^2}$$

9. $e^{(x+y)} = \ln(x^2 + y^2) + \sin x + \tan x$

$$e^{(x+y)}(1 + y') = \frac{2x + 2y\,y'}{x^2 + y^2} + \cos x + \sec^2 x$$

$$e^{(x+y)} + e^{(x+y)}y' = \frac{2x}{x^2 + y^2} + \frac{2y\,y'}{x^2 + y^2} + \cos x + \sec^2 x$$

$$e^{(x+y)}y' - \frac{2y\,y'}{x^2 + y^2} = \frac{2x}{x^2 + y^2} + \cos x + \sec^2 x - e^{(x+y)}$$

$$y'(e^{(x+y)} - \frac{2y}{x^2 + y^2}) = \frac{2x}{x^2 + y^2} + \cos x + \sec^2 x - e^{(x+y)}$$

$$y' = \frac{\frac{2x}{x^2 + y^2} + \cos x + \sec^2 x - e^{(x+y)}}{(e^{(x+y)} - \frac{2y}{x^2 + y^2})}$$

10. $y^x = x^y$

$$\ln y^x = \ln x^y$$

$$x \ln y = y \ln x$$

$$x\frac{y'}{y} + \ln y = y\frac{1}{x} + \ln x\, y'$$

$$\frac{x}{y}y' - y'\ln x = \frac{y}{x} - \ln y$$

$$y'(\frac{x}{y} - \ln x) = \frac{y}{x} - \ln y$$

$$y' = \frac{\frac{y}{x} - \ln y}{\frac{x}{y} - \ln x}$$

Inverse function

1. Trigonometric function

(1) $\frac{d}{x}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(2) $\frac{d}{x}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

(3) $\frac{d}{x}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

(4) $\frac{d}{x}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}$

(5) $\frac{d}{x}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad u > 1$

(6) $\frac{d}{x}\csc^{-1} u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad u > 1$

EXAMPLE 1: Prove that $\frac{d}{x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

Proof:

Let $y = \sin^{-1} x$

$$x = \sin y$$

$$\frac{d}{dx}x = \frac{d}{x}(\sin y)$$

$$1 = \cos y \frac{dy}{dx}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \sqrt{1 - \sin^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

EXAMPLE 2: Prove that $\frac{d}{x} \tan^{-1} x = \frac{1}{1 + x^2}$

Proof:

Let $y = \tan^{-1} x$

$$x = \tan y$$

$$\frac{d}{dx} x = \frac{d}{x} (\tan y)$$

$$1 = \sec^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} \qquad\qquad 1 + \tan^2 y = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

EXAMPLE 3: If $y = \tan^{-1}(x^2 - x)$ find $\frac{dy}{dx}$

Sol:

$$y' = \frac{1}{1 + u^2} = \frac{2x - 1}{1 + (x^2 - x)^2}$$

EXAMPLE 4: If $y = \sin^{-1} \ln(x)$ find y'

Sol:

$$y' = \frac{1}{\sqrt{1-u^2}} = \frac{1/x}{\sqrt{1-(\ln x)^2}}$$

EXAMPLE 5: $y = e^{\tan^{-1}(3x)}$ find y'

Sol:

$$y' = e^{\tan^{-1}(3x)} \frac{3}{1+(3x)^2}$$

EXAMPLE 6: $y = \ln(e^{\sin^{-1} x} - \tan^{-1} x)$

Sol:

$$y' = \frac{e^{\sin^{-1} x} \frac{1}{\sqrt{1-x^2}} - \frac{1}{1+x^2}}{e^{\sin^{-1} x} - \tan^{-1} x}$$

EXAMPLE 7: If $y = \cot^{-1} 2/x + \tan^{-1} x/2$

Sol:

$$y' = \frac{-(-2/x^2)}{1+(2/x)^2} + \frac{1/2}{1+(x/2)^2}$$

$$y' = \frac{2/x^2}{1+4/x^2} + \frac{1/2}{1+(x/2)^2}$$

EXAMPLE 8: If $y = \sin^{-1}(\frac{x-1}{x+1})$ find y'

Sol:

$$y' = \frac{\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}}{\sqrt{1-(\frac{x-1}{x+1})^2}} = \frac{\frac{x+1-x+1}{(x+1)^2}}{\sqrt{1-(\frac{x-1}{x+1})^2}}$$

$$y' = \frac{2/(x+1)^2}{\sqrt{1-(\frac{x-1}{x+1})^2}}$$

EXAMPLE 9: If $y = \sec^{-1}(5x)$ find y'

Sol:

$$y' = \frac{5}{5x\sqrt{25x^2-1}}$$

EXAMPLE 10: If $y = x \cdot \ln \sec^{-1} x$ find $\frac{dy}{dx}$

Sol:

$$y' = x\frac{\frac{1}{x\sqrt{x^2-1}}}{\sec^{-1} x} + \ln(\sec^{-1} x)$$

$$y' = \frac{1}{\sqrt{x^2-1}\ \sec^{-1} x} + \ln(\sec^{-1} x)$$

EXAMPLE 11: $y = 3^{\sin^{-1}(2x)} \Rightarrow$ find y'

Sol:

$$y' = \ln 3\ \ 3^{\sin^{-1}(2x)} \frac{2}{\sqrt{1-4x^2}}$$

<u>Hyperbolic function</u>

If u is any differentiable function of x

1. $\frac{d}{x}\sinh u = \cosh u \frac{du}{dx}$

2. $\frac{d}{x}\cosh u = \sinh u \frac{du}{dx}$

3. $\frac{d}{x}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$

4. $\frac{d}{x}\coth u = -\operatorname{csc h}^2 \frac{du}{dx}$

5. $\frac{d}{x}\operatorname{sec h}u = -\operatorname{sec h}u \tanh u \frac{du}{dx}$

6. $\frac{d}{x}\operatorname{csc h}u = -\operatorname{csc h}u \coth u \frac{du}{dx}$

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following function:

1. $y = \coth(\tan\ x)$

Sol:

$$y' = -\operatorname{csch}^2(\tan x)\sec^2 x$$

2. $y = \sin^{-1}(\tanh x)$

Sol:

$$y' = \frac{\operatorname{sec h}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sec h}^2 x}{\sqrt{\operatorname{sec h}^2 x}} = \operatorname{sec h} x$$

3. $y = \ln \tanh\ x/2$

Sol:

$$y' = \frac{\operatorname{sec h}^2 x/2\ \cdot(1/2)}{\tanh\ x/2} = \frac{1}{2}\frac{\frac{1}{\cosh^2 x/2}}{\frac{\sinh\ x/2}{\cosh x/2}}$$

$$y' = \frac{1}{2\sinh x/2\cdot\cosh x/2} = \frac{1}{\sinh x} = \operatorname{csch x}$$

4. $y = x\sinh 2x - \frac{1}{2}\cosh 2x$

Sol:

$$y' = x\cosh 2x\cdot 2 + \sinh 2x - \tfrac{1}{2}\sinh 2x\cdot 2$$

$$y' = 2x\cosh 2x$$

5. $y = \text{sech}^3 x$

Sol:

$$y' = 3\text{sech}^2 x(-\text{sech}x \cdot \tanh x)$$

$$y' = -3\text{sech}^3 x \ \tanh x)$$

6. $y = \text{csch}^2 x$

Sol:

$$y' = 2\csc\text{h}x(-\csc\text{h}x\coth x)$$

$$y' = -2\text{csch}^2 x\coth x$$

EXAMPLE 2: Prove that $\frac{d}{x}\tanh u = \sec\text{h}^2 u\frac{du}{dx}$

Sol:

$$\frac{d}{x}\tanh u = \frac{d}{x}(\frac{\sinh u}{\cosh u})$$

$$= \frac{\cosh u \cosh u \frac{du}{dx} - \sinh u \sinh u \frac{du}{dx}}{\cosh^2 u}$$

$$= \frac{(\cosh^2 u - \sinh^2 u) \frac{du}{dx}}{\cosh^2 u} = \frac{1}{\cosh^2 u}\frac{du}{dx}$$

$$\frac{d}{x}\tanh u = \sec\text{h}^2 u\frac{du}{dx}$$

EXAMPLE 3: Prove that $\frac{d}{x}\sec\text{h}u = -\sec\text{h}u\tanh u\frac{du}{dx}$

Sol:

$$= \frac{d}{dx}\frac{1}{\cosh u} = -\frac{1}{\cosh^2 u}\sinh u\frac{du}{dx}$$

$$= -\operatorname{sec} hu \tanh u \frac{du}{dx}$$

EXAMPLE 4: Show that the functions

$$x = -\frac{2}{\sqrt{3}} \sinh(t/\sqrt{3}) \qquad y = \frac{1}{\sqrt{3}} \sinh(t/\sqrt{3}) + \cosh(t/\sqrt{3})$$

Taken together, satisfy the differential equation

I. $$\frac{dx}{dt} + 2\frac{dy}{dt} + x = 0$$

II. $$\frac{dx}{dt} - \frac{dy}{dt} + y = 0$$

Proof: I

$$\frac{dx}{dt} = -\frac{2}{\sqrt{3}} \cosh(t/\sqrt{3}) \frac{1}{\sqrt{3}}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{3}} \cosh(t/\sqrt{3}) \frac{1}{\sqrt{3}} + \sinh(t/\sqrt{3}) \frac{1}{\sqrt{3}}$$

I. $$\frac{-2}{3} \cosh(t/\sqrt{3}) + 2\frac{1}{3} \cosh(t/\sqrt{3}) + \frac{2}{\sqrt{3}} \sinh(t/\sqrt{3}) + \frac{-2}{\sqrt{3}} \sinh(t/\sqrt{3}) = 0$$

II. $$\frac{-2}{3} \cosh(t/\sqrt{3}) - \frac{1}{3} \cosh(t/\sqrt{3}) - \frac{1}{\sqrt{3}} \sinh(t/\sqrt{3}) + \frac{1}{\sqrt{3}} \sinh(t/\sqrt{3}) + \cosh(t/\sqrt{3}) = 0$$