Integration

Integration is the reversal of differentiation hence functions can be integrated by indentifying the anti-derivative.

<u>Terminology</u>

Indefinite and **Definite** integrals
There are two types of integrals: Indefinite and Definite.
Indefinite integrals are those with no limits and definite integrals have limits.
When dealing with indefinite integrals you need to add a constant of integration.
For example, if integrating the function f(x) with respect to x:

$$\int f(x)\ dx = g(x) + C$$

where $g(x)$ is the integrated function.
C is an arbitrary constant called the constant of integration.
dx indicates the variable with respect to which we are integrating, in this case, *x*.
The function being integrated, f(*x*), is called the **integrand**.

The Rule

1) **Constant Rule** $\boxed{\int a\,dx = ax + c}$ where a is constant

EXAMPLE:

1. $\int 3\,dx = 3x + c$

2. $\int 4\,dy = 4y + c$

3. $\int \frac{7}{2} dz = \frac{7}{2} z + c$

2) **Sum Rule** $\boxed{\int (f \pm g)\,dx = \int f\ \ dx \pm \int g\,dx}$

EXAMPLE: **1.** $\int 3\,dx + 4dy = \int 3\,dx + \int 4dy = 3x + 4y + c$

3) The Power Rule $n \neq -1$

$$\int a x^n du = a\frac{x^{n+1}}{n+1} + c$$

EXAMPLE: **1.** $\int 4x^5 dx = 4\frac{x^6}{6} + c = \frac{2}{3}x^6 + c$

2. $\int 10x^{-5} dx = 10\frac{x^{-4}}{-4} + c = -\frac{5}{2}x^{-4} + c$

4) The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval *I* and f is continuous on *I*, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

EXAMPLES: **1.** $\int (x+1)^3 dx = \frac{(x+1)^4}{4} + c$

2. $\int (x^2+1)^4\, 2x\, dx = \frac{(x^2+1)^5}{5} + c$

3. $\int (x^3-4)^7\, 3x^2\, dx = \frac{(x^3-4)^8}{8} + c$

4. $\int (x^2-2x-4)^7\, (2x-2)\, dx = \frac{(x^2-2x-4)^8}{8} + c$

5. $\int (x^2+2)^5\, x\, dx = \frac{1}{2}\int (x^2+2)^5\, 2x\, dx$

$$= \frac{1}{2}\frac{(x^2+2)^6}{6} + c$$

6. $\int (x^2+2x+3)^7\, (x+1)\, dx = \frac{1}{2}\int (x^2+2x+3)^7\, 2(x+1)\, dx$

$$=\frac{1}{2}\int(x^2+2x+3)^7(2x+2)\,dx=\frac{1}{2}\frac{(x^2+2x+3)^8}{8}$$

Root function integral

EXAMPLES: **1.** $\int 2x\sqrt{x^2-3}\,dx=\int(x^2-3)^{1/2}\,2x\,dx$

$$\frac{(x^2-3)^{3/2}}{3/2}+c$$

$$\frac{2}{3}(x^2-3)^{3/2}+c$$

2. $\int x\sqrt{3x^2+1}\,dx=\frac{1}{6}\int(3x^2+1)^{1/2}\,6x\,dx$

$$=\frac{2}{3}\frac{(3x^2+1)^{3/2}}{6}+c=\frac{1}{9}(3x^2+1)^{3/2}+c$$

3. $\int\frac{x-1}{\sqrt{x^2-2x-3}}\,dx=\frac{1}{2}\int(x^2-2x-3)^{-1/2}(2x-2)\,dx$

$$=\frac{1}{2}\frac{(x^2-2x-3)^{1/2}}{1/2}+c$$

$$=\sqrt{(x^2-2x-3)}+c$$

4. $\int(x^3-1)^2\,x\,dx=\int(x^6-2x^3+1)\,x\,dx$

$$=\int(x^7-2x^4+x)\,dx$$

$$=\frac{x^8}{8}-\frac{2}{5}x^5+\frac{1}{2}x+c$$

5. $\int\frac{(\sqrt{x}-1)^2}{\sqrt{x}}\,dx$

$$=\int\frac{(\sqrt{x})^2-2\sqrt{x}+1}{\sqrt{x}}\,dx$$

$$=\int(\frac{x}{\sqrt{x}}-2\frac{\sqrt{x}}{\sqrt{x}}+\frac{1}{\sqrt{x}})\,dx$$

$$=\int(\sqrt{x}-2+\frac{1}{\sqrt{x}})\,dx$$

$$=\int(x^{1/2}\,dx-\int 2\,dx+\int x^{-1/2}dx$$

$$=\frac{x^{3/2}}{3/2}-2x+\frac{x^{1/2}}{1/2}+c=\frac{2}{3}x^{3/2}-2x+2x^{1/2}+c$$

Or $\int\frac{(\sqrt{x}-1)^2}{\sqrt{x}}dx=2\int(\sqrt{x}-1)^2\frac{1}{2\sqrt{x}}dx=2\frac{(\sqrt{x}-1)^3}{3}+c$

6. $\int\frac{1}{x}\sqrt{x^2-x^3}\,dx$

$$=\int\frac{1}{x}\sqrt{x^2(1-x)}\,dx$$

$$=\int\frac{x}{x}\sqrt{(1-x)}\,dx$$

$$=\int\sqrt{(1-x)}\,dx$$

$$=-\int-(1-x)^{1/2}\,dx$$

$$=-\frac{(1-x)^{3/2}}{3/2}+c=-\frac{2}{3}(1-x)^{3/2}+c$$

7. $\int\frac{dx}{(4x^2-12x+9)^{3/2}}\,dx$

$$=\int(4x^2-12x+9)^{-3/2}\,dx$$

$$=\int((2x-3)^2)^{-3/2}\,dx$$

$$=\int(2x-3)^{-3}\,dx$$

$$=\frac{1}{2}\int(2x-3)^{-3}\,2\,dx$$

$$=\frac{1}{2}\frac{(2x-3)^{-2}}{-2}+c=-\frac{1}{4}(2x-3)^{-2}+c$$

H.W **Evaluate**

1. $=\int\sqrt{x^2-x^4}\,dx$

2. $\int(x^2+1)^2(x+2)\,dx$

3. $\int\frac{2x-4}{\sqrt{x^2-4x+1}}\,dx$