Transcendental Function Integral 2

Invers function

1. If $y = \sin^{-1} u \quad \rightarrow \quad y' = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$

$$\boxed{\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} x + c}$$

$$\boxed{\int \frac{du}{\sqrt{a^2-u^2}} = \begin{cases} \sin^{-1} \frac{x}{a} + c \\ -\cos^{-1} \frac{x}{a} + c \end{cases}}$$

2. $y = \tan^{-1} u \quad \rightarrow \quad y' = \frac{1}{1+u^2} \frac{du}{dx}$

$$\boxed{\int \frac{1}{1+u^2} du = \tan^{-1} x + c}$$

$$\boxed{\int \frac{1}{a^2+u^2} du = \begin{cases} \frac{1}{a} \tan^{-1} \frac{u}{a} \\ -\frac{1}{a} \cot^{-1} \frac{u}{a} \end{cases}}$$

3. $y = \sec^{-1} u \quad \rightarrow \quad y' = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$

$$\boxed{\int \frac{1}{u\sqrt{u^2-1}} du = \sec^{-1}|u| + c}$$

$$\boxed{\int \frac{1}{u\sqrt{u^2-a^2}} du = \begin{cases} \frac{1}{a} \sec^{-1} \left|\frac{u}{a}\right| \\ -\frac{1}{a} \csc^{-1} \left|\frac{u}{a}\right| \end{cases}}$$

EXAMPLES 1: $\int \frac{x\,dx}{\sqrt{1-x^4}}\,dx$

$u^2 = x^4$

$u = x^2$

$du = 2x\,dx$

Sol:

$$\frac{1}{2}\int \frac{2x\,dx}{\sqrt{1-x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1} u + c$$

$$= \frac{1}{2}\sin^{-1} x^2 + c$$

2: $\int \frac{x^2}{\sqrt{3-x^6}}\,dx$

$u^2 = x^6$

$u = x^3$

$du = 3x^2\,dx$

Sol:

$$= \frac{1}{3}\int \frac{3x^2}{\sqrt{3-x^6}}\,dx$$

$$= \frac{1}{3}\int \frac{du}{\sqrt{3-u^2}}$$

$$= \frac{1}{3}\sin^{-1}\frac{u}{\sqrt{3}} + c$$

$$= \frac{1}{3}\sin^{-1}\frac{x^3}{\sqrt{3}} + c$$

3: $\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{2x}{1+x^4}\,dx$

$$= \frac{1}{2}\tan^{-1} x^2 + c$$

4: $\int \frac{x}{4+x^4}\,dx = \frac{1}{2}\left(\frac{1}{2}\tan^{-1}\frac{x^2}{2}\right) + c = \frac{1}{4}\tan^{-1}\frac{x^2}{2} + c$

5: $\int \frac{x^3}{7+x^8}\,dx$

$u^2 = x^8$

$u = x^4$

$du = 4x^3\,dx$

$$= \frac{1}{4}\int \frac{4x^3}{7+x^8}\,dx$$

$$= \frac{1}{4}\int \frac{du}{7+u^2}$$

$$= \frac{1}{4}\frac{1}{\sqrt{7}}\tan^{-1}\tfrac{u}{\sqrt{7}} + c = \frac{1}{4\sqrt{7}}\tan^{-1}\tfrac{x^4}{\sqrt{7}} + c$$

6:

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx$$

$$u^2 = (e^x)^2$$

$$u = e^x$$

$$du = e^x\,dx$$

$$= \int \frac{du}{1+u^2} = \tan^{-1} u + c$$

$$= \tan^{-1}(e^x) + c$$

7:

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx$$

$$= \sin^{-1}(e^x) + c$$

8:

$$\int \frac{dx}{x(1+\ln x)} = \int \frac{x^{-1}dx}{(1+\ln x)}$$

$$= \int \frac{1/x\,dx}{(1+\ln x)}$$

$$= \ln(1+\ln x) + c$$

9:

$$\int \frac{dx}{x(1+\ln x)^2} = \int (1+\ln x)^{-2}\,\frac{1}{x}dx$$

$$= \frac{(1+\ln x)^{-1}}{-1}$$

10:

$$\int \frac{dx}{x(1+(\ln x)^2)} = \int \frac{1/x\,dx}{1+(\ln x)^2}$$

$$u^2 = (\ln x)^2$$

$$u = \ln x$$

$$du = \frac{1}{x}dx$$

$$= \int \frac{du}{1+u^2} = \tan^{-1} u + c$$

$$= \tan^{-1}(\ln x)^2 + c$$

11: $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx = \sin^{-1}(\tan x) + c$

12: $\int \frac{2}{2x\sqrt{(2x)^2-1}}\, dx = \sec^{-1}(2x) + c$

13: $\int \frac{2}{\sqrt{x}(1+x)}\, dx = 4\int \frac{1/2\sqrt{x}\, dx}{1+(\sqrt{x})^2}$

$= 4\tan^{-1}\sqrt{x} + c$

14: $2\int \frac{\cos x\, dx}{1+(\sin x)^2} dx = 2\tan^{-1}(\sin x) + c$

15: $= \int \tan^{-1} x \frac{dx}{1+x^2} = \frac{(\tan^{-1} x)^2}{2} + c$

16: $= \int e^{\sin^{-1} x} \frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1} x} + c$

17: $= \int \frac{\sin^{-1} x\, dx}{\sqrt{1-x^2}} = \frac{(\sin^{-1} x)^2}{2} + c$

18: $= \int \frac{\tan^{-1}(x)\, dx}{1+x^2} = \frac{(\tan^{-1}(x))^2}{2} + c$

19: $= \int e^{\tan^{-1} x} \frac{dx}{1+x^2} = e^{\tan^{-1} x} + c$

20: $= \int \frac{dx}{x\sec^{-1}(x)\sqrt{x^2-1}} = \int \frac{\frac{dx}{x\sqrt{x^2-1}}}{\sec^{-1}(x)}$

$= \ln\left|\sec^{-1} x\right| + c$

21: $\int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \left(\frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}\right)\,dx$

$= \sin^{-1} x - \frac{1}{2}\int 2x(1-x^2)^{-1/2}\,dx$

$= \sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + c$

22: $\int \frac{1+x}{1+x^2}\,dx = \int \left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right)\,dx$

$= \tan^{-1} x + \frac{1}{2}\ln\left|1+x^2\right| + c$

23: $\int e^{\ln(\tan^{-1} x)}\,\frac{dx}{1+x^2}$

$\int \tan^{-1} x\,\frac{dx}{1+x^2}$

$= \frac{(\tan^{-1} x)^2}{2} + c$