Definition: A field is aset F with two operation, called addition $(+)$ and multiplacation $(\cdot)$ which satisfy the following:

1- if $x, y \in F$ then $x+y \in F$

2- if $x, y \in F$ then $x+y = y+x$

3- if $x, y, z \in F$ then $(x+y)+z = x+(y+z)$

4- $\exists$ an element $0 \in F$ s.t $x+0 = 0+x = x \quad \forall x \in F$

5- $\forall x \in F$, $\exists$ an element $x \in F$ s.t $x+(-x) = (-x)+x = 0$

6- if $x, y \in F$ then $x \cdot y \in F$

7- if $x, y \in F$ then $x \cdot y = y \cdot x$

8- if $x, y, z \in F$ then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

9- $\exists$ an element $1 \in F$ s.t $x \cdot 1 = 1 \cdot x = x$

10- $\forall x \in F$ then $\exists x^{-1} \in F$ s.t $x \cdot x^{-1} = x^{-1} \cdot x = 1$

11- if $x, y, z \in F$ $x \cdot (y+z) = x \cdot y + x \cdot z$

Examples: Show that $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are field.

Example: $(\mathbb{Z}, +, \cdot)$ is not field

because $\forall x \in \mathbb{Z}$, $\nexists x^{-1} \in \mathbb{Z}$ s.t

$$x \cdot x^{-1} = x^{-1} \cdot x = 1$$

since $2 \in \mathbb{Z}$ but $2^{-1} = \frac{1}{2} \notin \mathbb{Z}$

s.t $2 \cdot \frac{1}{2} = \frac{1}{2} \cdot 2 = 1$

then $(\mathbb{Z}, +, \cdot)$ is not field.

Definition: $(F, +, \cdot, <)$ is called order field if:

1- $(F, +, \cdot)$ is a field
2- $(F, <)$ is an order set
3- if $a \leqslant b$ then $a + c \leqslant b + c \quad \forall a, b, c \in F$
4- if $a, b \in F$, $a > 0$, $b > 0$ then $a \cdot b > 0$

Example: $(Q, +, \cdot, <)$ is an order field

Solution:

1- let $x, y \in Q$

$$x = \frac{a}{b}, \quad y = \frac{c}{d} \qquad a, b, c, d \in \mathbb{Z} \quad b, d \neq 0$$

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \in Q$$

2- if $x, y \in Q$

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} = \frac{bc + ad}{bd} = y + x$$

$\therefore \; x + y = y + x$

3- if $x, y, z \in Q$

$$x + (y + z) = \frac{a}{b}\left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} + \frac{fc + de}{df} = \frac{adf + bfc + bde}{bdf}$$

$$(x + y) + z = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{ad + bc}{bd} + \frac{e}{f} = \frac{adf + bcf + bde}{bdf}$$

$\therefore \; x + (y + z) = (x + y) + z$

4- $\exists\, 0 \in Q$ s.t $0 + \frac{a}{b} = \frac{a}{b} + 0 = \frac{a}{b}$

i.e $0 + x = x + 0 = x$

5- $\forall x \in Q \quad \exists -x \in Q$ s.t $x + (-x) = 0$

$$\frac{a}{b} + \left(\frac{-a}{b}\right) = 0$$

6- if $x, y \in Q$

$$x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \in Q$$

7- if $x, y \in Q$

$$x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = y \cdot x$$

$\therefore \; x \cdot y = y \cdot x$

8- if $x, y, z \in Q$

$$x \cdot (y \cdot z) = \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \frac{ace}{bdf}$$

$$(x \cdot y) \cdot z = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{ace}{bdf}$$

$\therefore \; x \cdot (y \cdot z) = (x \cdot y) \cdot z$

9- $1 \in Q$ s.t $1 \cdot \frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b}$ $\therefore \; 1 \cdot x = x \cdot 1 = x$

10- $\forall \; x \in Q$, $\exists \; x^{-1} \in Q$ s.t $x \cdot x^{-1} = 1$

i.e $\frac{a}{b} \cdot \frac{b}{a} = 1$

11- H.w

$\therefore \; (Q, +, \cdot)$ is a field.

2- $(Q, <)$ is an order set

3- if $x \leq y$ then $x + z \leq y + z \quad \forall \; x, y, z \in Q$

$$\frac{1}{4} < \frac{1}{2} \Rightarrow \frac{1}{4} + \frac{1}{3} \leq \frac{1}{2} + \frac{1}{3} \Rightarrow \frac{7}{12} < \frac{5}{6}$$

4- if $x, y \in Q$, $x > 0$, $y > 0$ then $x \cdot y > 0$

$$\frac{1}{2}, \frac{1}{4} \in Q \Rightarrow \frac{1}{8} > 0$$

$\therefore \; (Q, +, \cdot, <)$ is an order field.

Definition: Let $(F, +, \cdot, <)$ are order field then $(F, +, \cdot, <)$ is called complete order field if every subset $E \subseteq F$ which is bounded above has least upper bound in F.

Example: The real number is complete order field.

Completeness prporty of $\mathbb{R}$:

[Every non empty set of real number which is a bounded above has least upper bound in $\mathbb{R}$]

Example: The rational number is not complete order field.

Sol: Consider the set $S = \{x \in Q : x^2 < 2\}$

3 is upper bound of S

$\therefore$ S is bounded above

But S deos not has least upper bound in Q

$\therefore$ $(Q, +, \cdot, <)$ is not complete order field.

« The relation between field of real numbers and field of rational numbers »

<u>proposition</u> : Every order field contians the integer numbers.

<u>proof</u> : Let $(F, +, \cdot, <)$ be an order field and 0 is the addictive identity and 1 is the multiplicative identity

Now $1+1 \in F$, we claim that $1+1 \neq 1$
and $1+1 \neq 0$

suppose $1+1 = 1$

$$-1+(1+1) = -1+1$$
$$(-1+1)+1 = -1+1$$
$$0+1 = 0$$
$$1 = 0 \qquad \text{C!} \quad (0<1 \text{ since } F \text{ is an order field})$$

$\therefore\ 0<1 \quad , \quad 0+0 < 1+1$
$\qquad 0 < 1+1$

But $1+1=2$ and $2 \neq 1$

Similarly we can prove $3 = 1+1+1 \neq 2$

By induction $n = \underbrace{1+1+\cdots+1}_{n-\text{times}} \in F$

since F is a field and $n \in F \Rightarrow -n \in F$

$\therefore$ F contians the integer number.

<u>Corollary</u> : Every order field contians the field of rational numbers.

<u>proof</u>: since $n \in F$ (proposition above)

and F is a field $\Rightarrow \frac{1}{n} \in F$

Now $\frac{1}{n}, m \in F \Rightarrow \frac{1}{n} \cdot m \in F$

$\frac{m}{n} \in F$

$\therefore$ F contians the field of rational number

since $\mathbb{R}$ is an order field

then $Q \subseteq \mathbb{R}$.

Example : Show that the equation $x^2 = 2$ has no roots in the field rational numbers.

Solution : Suppose y is a rational number

$\ni y^2 = 2$

let $y = \frac{m}{n}$, $m, n \in \mathbb{Z}^+$ positive integer and $n \neq 0$

$(\frac{m}{n})^2 = 2 \Rightarrow \frac{m^2}{n^2} = 2$

$\therefore m^2 = 2n^2$

1- suppose m is even and n is odd

$m = 2K \Rightarrow m^2 = 4K^2$

$\therefore 4K^2 = 2n^2$

$2K^2 = n^2$

n^2 is even

but n is odd $\Rightarrow n^2$ is odd

$\therefore n^2$ is even and odd which is contraduction

2- suppose m is odd and n is even

$n = 2K \Rightarrow n^2 = 4K^2$

$m^2 = 2 \cdot 4K^2$

m^2 is even

but m is odd

$\Rightarrow m^2$ is odd

$\therefore m^2$ is even and odd which is contrduction

3- suppose m and n are odd

$m^2 = 2n^2$

$\therefore m^2$ is even

but m is odd

$\Rightarrow m^2$ is odd

$\therefore m^2$ is even and odd which is contraduction

$\therefore$ where is no rational number is satisfy the equation $x^2 = 2$

Corollary: There existe a unique positive real number such that $x^2 = 2$.

proof: Let $S = \{ x : x \in \mathbb{R} \wedge x \geq 0 , x^2 < 2 \}$

S is non empty since $0 \in S$

$\Rightarrow S$ is bounded above and $S \subseteq \mathbb{R}$

$\therefore$ By The completness of the real numbers.

There exist least upper bounded of S in $\mathbb{R}$.

Let $y_0 = \sup(S) \Longrightarrow y_0 > 0$

we claim $y_0^2 = 2$, suppose $y_0^2 \neq 2$

1- If $y_0^2 < 2$

Let $0 < h < 1$ and $h < \frac{2 - y_0^2}{2y_0 + 1}$

$(y_0 + h)^2 = y_0^2 + 2y_0 h + h^2 < y_0^2 + h(2y_0 + 1)$ $(h^2 < h)$

By hypothess $h < \frac{2 - y_0^2}{2y_0 + 1}$

$y_0^2 + h(2y_0 + 1) < 2$

$\Rightarrow (y_0 + h)^2 < 2 \Longrightarrow y_0 + h \in S$

$\therefore y_0$ is upper bounded of S and $y_0 < \underbrace{y_0 + h}_{\in S}$ C!

$\therefore$ It is not true that $y_0^2 < 2$

2- $y_0^2 > 2$

let $0 < K < 1$ and $K < \frac{y_0^2 - 2}{2y_0 + 1}$

$(y_0 - K)^2 = y_0^2 - 2y_0 K + K^2 \geq y_0^2 - 2y_0 K - K$ $(K^2 > -K)$

$\geq y_0^2 - K(2y_0 - 1)$

By hyp. $K < \frac{y_0^2 - 2}{2y_0 + 1}$

$\Rightarrow y_0^2 - K(2y_0 + 1) > 2 \Longrightarrow (y_0 - K)^2 > 2$

$\therefore y_0 - K$ is upper bounded of S and $y_0 - K < y_0$

since $y_0 = \sup(S)$

$\therefore y_0 - K < y_0$ C!.

$\therefore$ It is not true that $y_0^2 > 2$

Thus $\exists$ positive real number x s.t $x^2 = 2$

Archimedean Property:

If $a, b \in \mathbb{R}$ and $a > 0$ there is a positive integer n such that $n \cdot a > b$

proof: Let $X = \{ k \cdot a : k \in \mathbb{N} \}$, $X \neq \emptyset$, $X \subseteq \mathbb{R}$

suppose that the property is false

i.e $\forall n \in \mathbb{N}$ then $n \cdot a < b$

$\Rightarrow$ $k \cdot a < b$

$\therefore$ b is upper bound of X.

since $\mathbb{R}$ is complete and $X \subseteq \mathbb{R}$

then X has L.u.b in $\mathbb{R}$ is y

i.e $y = \sup(X)$

$\because$ $a > 0$ $\Rightarrow$ $y - a < y$

$\therefore$ $y - a$ is not l.u.b of X

let $\exists$ $m \in \mathbb{N}$ s.t $m \cdot a \in X$ and $y - a < m \cdot a$

$\Rightarrow$ $(m+1) a > y$

But $(m+1) a \in X$ C!

$\therefore$ $n \cdot a > b$

Corollary: If $\epsilon \in \mathbb{R}$ and $\epsilon > 0$ then there exist positive integer $n \ni \frac{1}{n} < \epsilon$

proof: put $p = 1$ and $a = \epsilon$

$\exists$ a positive integer n s.t $n \cdot \epsilon > 1$

[By Archimedean prp.]

$n \cdot \epsilon > 1 \quad \div$

$\epsilon > \frac{1}{n}$

$\therefore \frac{1}{n} < \epsilon$

The density of the rational numbers

Theorem: If $a \in \mathbb{R}$, $b \in \mathbb{R}$ s.t $a < b$ then $\exists\, r \in Q$ s.t $a < r < b$

proof: 1- suppose $0 < a < b$ and $b - a > 1$

Let $S = \{ n \in \mathbb{N} : n = n.1 > a \}$. $S \neq \emptyset$

By Arech. prop.

Let K be the smallest number in S

$K - 1 < a < K$ ...①   k اصغر عنصر موجود في S ولكن اكبر من a

since $1 < b - a < b - (K-1)$

$1 < b - K + 1$

$1 - 1 < b - K \Rightarrow 0 < b - K$

$\Rightarrow K < b$ ...②

from ① and ②

$\therefore a < K < b$

$\therefore K$ is a rational number between a and b

2- suppose $0 < b-a \leq 1$

by Archiemdean prop. $\exists$ a positive integer n

$\ni$ $n(b-a) > 1$

$\Rightarrow$ $nb - na > 1$

from ① and Arch. prop.

$\exists$ apositive integer K s.t $na < K < nb$

then $a < \frac{K}{n} < b$

$\therefore$ $\frac{K}{n}$ is arational number between a and b

3- $a < 0 < b$

0 is arational number between a and b

4- suppose $a < b < 0$

then $0 < -b < -a$

$\therefore$ $\exists\, r \in Q$ s.t $-b < r < -a$

$\Rightarrow$ $a < -r < b$

$\therefore$ $-r$ is arational number between a and b

Corollary:

If $a, b \in \mathbb{R}$ s.t $a < b$ then the set of all rational number between a and b is infinite

proof: since $a < b$ and $a, b \in \mathbb{R}$

By above theorem

$\exists\, r_1 \in Q$ s.t $a < r_1 < b$ also

$\exists\, r_2 \in Q$ s.t $r_1 < r_2 < b$

arbitary $\exists\, r_n \in Q$ s.t $r_{n-1} < r_n < b$

Then we get aset of rational number which is infinite

Theorem : (The density of irrational number)
If $a, b \in \mathbb{R}$ s.t $a < b$ then there is s irrational number s.t $a < s < b$.

proof: suppose that the theorem is false

$\therefore \forall s \in \mathbb{R}$ and $a < s < b$, s is rational number

$\therefore a+\sqrt{2} < s+\sqrt{2} < b+\sqrt{2}$

$\because s+\sqrt{2}$ is irrational number

Thus there is no rational number between $a+\sqrt{2}$ and $b+\sqrt{2}$

which is contraduction with (density of rational number)

$\therefore a < s < b$ and s is irrational number.

Corrollary: If $a, b \in \mathbb{R}$ and $a < b$, the set of irrational numbers between a and b is infinite

proof: since $a < b$ and $a, b \in \mathbb{R}$

$\therefore$ by theorem of density of irrational numbers

$\exists\ s_1 \in Q^c$ s.t $a < s_1 < b$ also

$\exists\ s_2 \in Q^c$ s.t $a < s_2 < s_1$

arbitary $\exists\ S_n$ s.t $a < s_n < s_{n-1}$

Then we get a set of irrational number which is infinite.