Chapter Two

The Metric Spaces

Definition:

Let X be a non-empty set and $d: X \times X \longrightarrow R$ is called the distance function satisfy the following conditions:

a) $d(x,y) \geq 0$, for all $x,y \in x$
b) $d(x,y) = 0$ iff $x=y$
c) $d(x,y) = d(y,x)$ for all $x,y \in x$
d) $d(x,y) \leq d(x,z) + d(z,y)$ (Triangle inequality) then (x,d) is called metric space.

Example:

let $d:RxR \longrightarrow R$ defined by $d(x,y) = |x-y|$, for all $x,y \in R$ show that (R,d) is a metric space

Sol:

1) $d(x,y) = |x-y| > 0$, for all $x,y \in R$ (By def. of absolutely value)
2) $d(x,y) = 0$
   $\Leftrightarrow |x-y| = 0$
   $\Leftrightarrow x-y = 0$
   $\Leftrightarrow x = y$
3) $d(x,y) = | x-y |$
   $= |-(y-x)|$
   $= |-1| . |y-x|$
   $= |y-x|$
   $= d(y,x)$
4) $d(x,y) = |x-y|$
   $= |x-z+z-y|$
   $\leq |x-z|+|z-y|$
   $\leq d(x,y) + d(z,y)$

∴ (R,d) is a metric space

<u>Lemma 2.1</u> : (Cauchy- Schwarz inequality)

For any real numbers $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ we have

$(a_1 b_1 + \dots + a_n b_n) \le \sqrt{a_1^2 + \cdots + a_n^2} \, . \sqrt{b_1^2 + \cdots + b_n^2}$

<u>Lemma 2.2</u> : Minkowski inequality

For any real numbers $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ we have

$$\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + \cdots + (a_n + b_n)^2} \le \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2}$$

Example :

Let $d: R^n x R^n \longrightarrow R$ defined by

$d((x_1,x_2, \dots, x_n), (y_1,y_2, \dots y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$

sol:

1) $(x_i - y_i)^2 \ge 0, \forall\, i = 1,2, \dots, n$

$$\Rightarrow \sum_{i=1}^{n} (x_i - y_i)^2 \ge 0$$

$$\Rightarrow \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \ge 0$$

$$\Rightarrow d(x, y) \ge 0, \forall\, x, y \in R^n$$

2) $d(x, y) = 0, \forall\, x, y \in R^n$

$$\Leftrightarrow \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = 0$$

$$\Leftrightarrow \sum_{i=1}^{n}(x_i - y_i)^2 = 0$$

$\Leftrightarrow (x_i,y_i)^2 = 0$, $\forall\, i = 1,2, \dots, n$

$\Leftrightarrow x_i = y_i$, $\forall\, i = 1,2, \dots, n$

$\Leftrightarrow (x_1,x_2,\dots, x_n) = (y_1,y_2,\dots, y_n)$

$\Leftrightarrow x = y$

3) $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$

$= \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2}$

$= d(y,x)$

4) let $x= (x_1,\dots, x_n)$, $y= (y_1,\dots, y_n)$, $z= (z_1,\dots, z_n) \in R$

$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$

$= \sqrt{\sum_{i=1}^{n}(x_i - z_i + z_i - y_i)^2}$

$\leq \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} + \sqrt{\sum_{i=1}^{n}(z_i - y_i)^2}$

[By minkowski inequality]

$\leq d(x,z) + d(z,y)$

$\therefore d \text{ is metric on } R^n$

$\therefore (R^n, d)$ is a metric space which is called n-dimensional Euclidean space

Example:

Let $d: R^2 x R^2 \rightarrow R$ defined by $d(x,y)= |x_1-y_1| + |x_2-y_2|$ where $x= (x_1, x_2)$ and $y=(y_1,y_2)$

Sol:

1) $\because |x_1 - y_1| + |x_2 - y_2| \geq 0$

$\therefore d(x,y) \geq 0$

2) d(x,y) = 0

$\Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0$

$\Leftrightarrow |x_1 - y_1| = 0 \; and \; |x_2 - y_2| = 0$

$\Leftrightarrow x_1 = y_1 \; and \; x_2 = y_2$

$\Leftrightarrow$ x = y

3) d(x,y) $= |x_1 - y_1| + |x_2 - y_2|$

$= |-(y_1 - x_1)| + |-(y_2 - x_2)|$

$= |-1||y_1 - x_1| + |-1||y_2 - x_2|$

$= |y_1 - x_1| + |y_2 - x_2|$

= d(y,x)

4) let $z = (z_1, z_2) \in R^2$

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

$$= |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2|$$
$$\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2|$$
$$\leq (|x_1 - z_1| + |x_2 - z_2|) + (|z_1 - y_1| + |z_2 - y_2|)$$
$$\leq d(x,z) + d(z,y)$$

$\therefore d \; is \; a \; metric \; on \; R^2$

$\therefore (R^2, d) is \; a \; metric \; space$

Example:

Let X be a non-empty set defined d: X*X → R by $d(x,y) = \begin{cases} 0 & if \; x = y \\ 1 & if \; x \neq y \end{cases}$

Sol:

1) $d(x,y) \geq 0$
2) $d(x,y) = 0 \; iff \; x = y$
3) $d(x,y) = d(y,x)$

   1 = 1 if $x \neq y$

   0 = 0 if $x = y$
4) $d(x,y) \leq d(x,z) + d(z,y)$

1) $if\ x \neq y \neq z$

$1 \leq 1 + 1$

2) $if\ x \neq y, x = z\ \&\ y \neq z$

$1 \leq 0 + 1$

3) $if\ x \neq y, x \neq z\ \&\ y = z$

$1 \leq 1 + 0$

4) $x = y\ \&\ x \neq z\ , y \neq z$

$0 < 1 + 1$

5) $x = y = z$

$0 \leq 0 + 0$

Exc. :

Let $c[a,b] = \{f: [a,b] \to \mathbb{R}\ be\ a\ cont.function\}, define\ d: c[a,b] * c[a,b] \to \mathbb{R}\ as\ d(f,g) = \int_a^b |f(x) - g(x)| dx\ \ show\ that\ (c[a,b], d) is\ M.s.$

Exc. :

Let $x = R^2$ we define $d: R^2 * R^2 \to R$ by $d(x,y) = max.\{|x_1 - y_1|, |x_2 - y_2|\}$ $(R^2, d) a\ metric\ space$?

Definition:

Let (x,d) be a metric space and let $x_0 \in X, r \in R, r > 0$

$B_r(x_0) = \{x \in X: d(x, x_0) < r\}$

Is called a ball of radius r and center x_0.

(Neighborhood of x_0 with radius r)

$D_r(x_0) = \{x \in X: d(x, x_0) \leq r\}$ is called disk with radius r and center x_0.

Example:

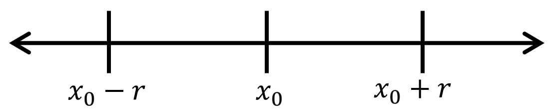
Let (R,d) be a metric space where $d(x,y) = |x - y|, \forall\, x, y \in R$

$B_r(x_0) = \{x \in R : d(x_0, x) < r\}$

$= \{x \in R : |x - x_0| < r\}$

$= \{x \in R : x_0 - r < x < x_0 + r\}$

$= (x_0 - r, x_0 + r)$

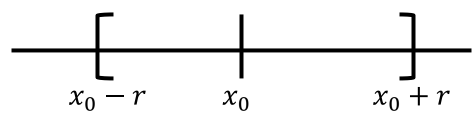


$D_r(x_0) = \{x \in R : d(x, x_0) \leq r\}$

$= \{x \in R : |x - x_0| \leq r\}$

$= \{x \in R : x_0 - r \leq x \leq x_0 + r\}$

$= [x_0 - r, x_0 + r]$



Example:

Let (R^2, d) be a metric space where $d: R^2 * R^2 \to R$ s.t. d is a usual distance

$B_r(x_0) = \{x \in R^2 : d(x, x_0) < r\}$

$= \{x \in R^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} < r\}$

$= \{x \in R^2 : (x - x_0)^2 + (y - y_0)^2 < r^2\}$

$D_r(x_0) = \{x \in R^2 : d(x, x_0) \leq r\}$

$= \{x \in R^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} \leq r$

$= \{x \in R^2 : (x - x_0)^2 + (y - y_0)^2 \leq r$

Example:

Let (R^n, d) be a metric space where $d: R^n * R^n \to R$ s.t. d is a usual distance on R^n

$B_r(x_0) = \{x \in R^n: d(x, x_0) < r\}$

$= \{x = (x_1, x_2, \dots, x_n) \in$
$R^n: \sqrt{(x_1 - x_{02})^2 + (x_2 - x_{02})^2 + \cdots + (x_n - x_{0n})^2} < r$

$= \{(x_1, x_2, \dots, x_n) \in R^n: (x_1 - x_{01})^2 + \cdots + (x_n - x_{0n})^2 < r^2\}$

Where $x_0 = (x_{01}, x_{02}, \dots, x_{0n})$

$D_r(x_0) = (Exc.)$

Define:

Let (x,d) be a metric space and $A \subseteq X$, an element $P \in A$ is called interier point if $\exists B_r(p) s.t. B_r(p) \subseteq A$, and all interier points of A denoted by A^0

Ex. : let (R, d) be a metric space where $A = (0,1), B = [-1,1], c = z$

Find A^0, B^0, C^0

Sol: $A^0 = (0,1), B^0 = (-1,1) and\ C^0 = \phi$

$A^0\ \forall x \in A \to (x{-}\in, x{+}\in) \subseteq A$

$B^0\ \forall\ x \in B \to \exists \in > 0\ s.t. (x{-}\in, x{+}\in) \subseteq B$

$1 \in B, \nexists \in > 0\ s.t. (1{-}\in, 1{+}\in) \subseteq B$

$-1 \in B, \nexists \in > 0\ s.t. (-1{-}\in, -1{+}\in) \subseteq B$

$C^0, \forall\ x \in C \to \exists \in > 0\ s.t. (x{-}\in, x{+}\in) \nsubseteq C$

Definition:

Let (x,d) be a metric space and $A \subseteq X$, A is called an open set if $\forall\ P \in A$ there exists $r > 0$ $(r \in R)$ such that $B_r(p) \subseteq A$.

i.e. A is open set iff $A^0 = A$.

Ex. : let (R,d) be a metric space, which of the following sets is open: $A = (0,1) is\ open\ set\ \ A^0 = (0,1) = A$.

Theorem 2.1:

Every ball (neighborhood) is an open set proof: $B_r(x_0) = \{x \in X: d(y, x_0) < r\}$

Let $y \in B_r(x_0) \to d(y, x_0) = r_1 < r$ take $\in = r - r_1 > 0$

T.P. $B_\in(y) \subseteq B_r(x_0)$

Let $z \in B_\in(y) \quad T.P. \quad z \in B_r(x_0)$

$d(z, y) < \in \quad T.P. \quad d(z, x_0) < r$

$d(z, x_0) \leq d(z, y) + d(y, x_0)$

$< \in + r_1$

$< r - r_1 + r_1$

$< r$

$\therefore d(z, x_0) < r \to z \in B_r(x_0)$

$\therefore B_\in(y) \subseteq B_r(x_0)$

Hence every point of $B_r(x_0) is\ an\ interier\ point$.

$\therefore B_r(x_0) is\ an\ open\ set$.

Remark: every open interval in R is an open set

Ex. : $(a, \infty), (-\infty, a), (a, c)$ ore open sets.

Sol: $\forall\ b \neq a, \exists d = |b - a|$

s.t. $(b - \in, b + \in) \subset (a, \infty)$

$\therefore (a, \infty)^0 = (a, \infty)$

$\therefore (a, \infty) is\ open\ set.$

Ex. : is $A = [a, b) open\ set$

Sol: for all $x \in (a, b) \rightarrow \exists n\ s.t. \frac{1}{n} < \in (x - \in, x + \in) \subseteq (a, b)$

but for all ball $(a - \in, a + \in) \nsubseteq [a, b)$

$\therefore A^0 = (a, b)$

$\therefore A^0 \neq A$

$\therefore A\ is\ not\ open.$

Ex. :

$H = \{(x, y) \in R^2 : x \in R, y \geq 0\}.$

Is H open set in R^2?

Sol : for all $(x, y) \in H\ s.t. y > 0 \rightarrow \exists B_r(x, y) s.t. B_r\big((x, y)\big) \subseteq H$ but if $y = 0$ and $x \in R$

$\rightarrow B_r((x, y)) \nsubseteq H$

Exc.: show that $k = \{(x, y) \in R^2 : x \in R, y > 0\}$ is open subset of R^2.

Ex.: the set of rational is not open set since any interval in Q with center $\frac{p}{q} \in Q$ doesn't contain rationales only (by the density of irrational)

Theorem 2.2: For any collection $\{G_i\}_{iGI}$ of open sets then $U_{i \in I} G_i\ is\ open.$

Proof: let $x \in U_{i \in I} G_i$

$\rightarrow x \in G_k, for\ some\ k \in I$

Since $G_k\ is\ open\ set$

$\therefore x \text{ is an interier point of } G_k$

$\therefore i.e. \exists B_r(x) s.t. B_r(x) \subseteq G_k$

$\therefore B_r(x) \subseteq U_{i\in I} G_i \text{ is open set}$

Theorem 2-3:

The intersection of a finite number of open set is open.

Proof: let $u_1, u_2, \dots, u_n$ be a set of finite number of open set.

T.P. $\bigcap_{i=1}^{n} u_i \text{ is open}$

Let $x \in \bigcap_{i=1}^{n} u_i$

$\Longrightarrow x \in u_i, \forall\, i = 1,2, \dots, n$

$\because u_i \text{ is open }, \forall\, i = 1,2, \dots, n$

$\therefore \exists r_i > 0 \ s.t. \ \ B_{r_i}(x) \subseteq u_i, \quad \forall i = 1,2, \dots, n$

Take $r = \min.\{r_1, r_2, \dots, r_n\}$

$$B_r(x) \subseteq \bigcap_{i=1}^{n} u_i$$

$$\therefore \bigcap_{i=1}^{n} u_i \text{ is open.}$$

Remark: the intersection of infinite number of open set needn't be open, as the following example show:

Ex. : let (R, d) be a metric space

$$\forall n \in N, let\ A_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$$

$$\bigcap_{n} A_n = \{0\}$$

By Arch. Property.

If $\exists 0 \neq x, x > 0, \exists k \in N \ s.t. \frac{1}{k} < x$

$\therefore x \notin \left(\frac{-1}{k}, \frac{1}{k}\right)$

$\therefore x \notin \bigcap_{n} A_n$

By arch. Property.

If $0 \neq x, x < 0 \Longrightarrow -x > 0 \Longrightarrow \exists t \in N \quad s.t. \ \frac{1}{t} < -x$

$\Longrightarrow \frac{-1}{t} > x$

$\therefore x \notin \left(\frac{-1}{t}, \frac{1}{t}\right)$

$\Longrightarrow x \notin \bigcap_{n} A_n$

$\{0\}$ is not open since $\forall \in > 0$

$B_{\in}(0) = (-\in, \in) \not\subset \{0\}$

Proposition 2.4: let (x,d) be a metric space and $A \subseteq X$ then A is open iff A is a union of balls.

Proof: $(\Longrightarrow)$ *suppose that A is an open set.*

$\Longrightarrow \forall x \in A, \exists r_x > 0 \quad s.t. B_{r_x}(x) \subseteq A$

$\therefore \bigcup_{x \in A} B_{r_x}(x) \subseteq A$

$(\Longleftarrow) let\ A = \bigcup_{i \in A} B_i, B_i are\ balls$

$\because$ *every ball is an open set*

$\Longrightarrow \bigcup_{i \in A} B_i$ *is open set.* $\quad$ [*theorom* 2.2]

Def. : Two metrics d and $d_1$on the some set X are said to be equivalent, if every open set in (x,d) is open in (x,d_1).

Ex. : let (x,d) be a metric space and P be a function on $X * X$, defined by $p(x,y) = min.\{1, d(x,y)\}, \forall x,y \in X$

Sol:

1) $\because d(x,y) \geq 0$
   $\therefore min.\{1, d(x,y)\} \geq 0$
   $\Longrightarrow P(x,y) \geq 0$
2) $P(x,y) = 0$
   $\Leftrightarrow min.\{1, d(x,y)\} = 0$
   $\Leftrightarrow d(x,y) = 0$
   $\Leftrightarrow x = y$
3) $P(x,y) = \min\{1, \mathrm{d(x,y)}\}$
   $= \min\{1, d(y,x)\}$
   $= P(y,x), \forall x,y \in X$
4) Let $x,y,z \in X$, If at least one of say $d(x,y) \geq 1$

Then $P(x,y) = \min\{1, d(x,y)\} = 1$

$\therefore P(x,y) + P(y,z) \geq 1 \geq P(x,z)$

Also in case $d(x,y) < 1$ *and* $d(y,z) < 1$

$P(x,y) = \min\{d(x,y), 1\} = d(x,y)$

$P(y,z) = \min\{d(y,z), 1\} = d(y,z)$

$$\therefore P(x,y) + P(y,z) = d(x,y) + d(y,z)$$
$$\geq d(x,z) by\ triangle\ inequality \geq P(x,z)$$

$\therefore P(x,z) \le P(x,y) + (y,z).$

$\therefore P\ is\ a\ metric\ on\ X\ and\ (X,P)\ is\ a\ metric\ space.$

Now to show that P is equivalent to d.

T.P. every open set in (X,P) is open in (x,d)

Let G be any open subset of X in (X,P)

Let $x \in G \Longrightarrow \exists an\ open\ set$

$\{y \in X: P(x,y) < r\} \subseteq G$

$\therefore P(x,y) \le d(x,y), \forall\ x,y \in X$

$\therefore \{y \in X :: d(x,y) < r\} \subseteq \{y \in X: P(x,y) < r\} \subseteq G$

$\therefore G\ is\ open\ in\ (x,d)$

Hence every open set in (X,P) is open set in (x,d)

Next, let H be an open set in (x,d)$\Longrightarrow \forall\ x \in H, \exists\ and\ ball$

$y \in X: d(x,y) < r\} \subseteq H$

Let $r^1 = \min\{1,r\}, so\ r^1 \le r, then$

$\{y \in X: P(x,y) < r^1\} \subseteq \{y \in X: d(x,y) < r\} \subseteq H.$

$\therefore H\ is\ open\ set\ in\ (X,P)$

$\therefore Every\ open\ set\ in\ (x,d) is\ open\ in\ (X,P)$

Hence d and P are equivalent metrics.

<u>Ex. :</u> Let (x,d) be a metric space, and let

$$d^x(x,y) = \frac{d(x,y)}{1+d(x,y)}, \forall\ x,y \in X$$

Show that d^* is a metric on X equivalent to d.

<u>Sol. :</u> First to show d^* is a metric on X

1) $\because d(x, y) \geq 0 \Longrightarrow d^*(x, y) \geq 0$

2) $d^*(x, y) = 0$

$\Longleftrightarrow d(x, y) = 0$

$\Longleftrightarrow x = y \ since\ d\ is\ metric\ on\ X$

3) $d^*(x, y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d^*(y, y)$

4) For all $x, y, z \in X, we\ have$:

$$d^*(x, y) + d^*(y, z) = \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)}$$

$$\geq \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)}$$

$$\geq \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)}$$

$$\geq 1 - \frac{1}{1 + d(x, y) + d(y, z)}$$

$\because d\ is\ a\ metric\ on\ X$

$\therefore d(x, y) + d(y, z) \geq d(x, z)$

$\Longrightarrow 1 + d(x, y) + d(y, z) \geq 1 + d(x, z)$

$$\Longrightarrow \frac{1}{1 + d(x, y) + d(y, z)} \leq \frac{1}{1 + d(x, z)}$$

$$\Longrightarrow 1 - \frac{1}{1 + d(x, y) + d(y, z)}$$

$$\geq 1 - \frac{1}{1 + d(x, z)}$$

$\geq d^*(x, z)$

$\therefore d^*(x, z) \leq d^*(x, y) + d^*(y, z)$

<u>Exc. :</u> now, to show d and d^* are equivalent.

Let $B_R(x), r > 0, be\ any\ d - open\ ball\ and\ B_P(x)$

Be $d^* - open$ ball where $P = \frac{r}{1+r}$

T.P. $B_P(x) \subseteq B(x,r)$
Let $y \in B_P(x) \Longrightarrow d^*(x,y) < P$
$\Longrightarrow \frac{d(x,y)}{1+d(x,y)}, \frac{r}{1+r}$
$\Longrightarrow d(x,y) + rd(x,y) < r + rd(x,y)$
$\Longrightarrow d(x,y) < r$
$\Longrightarrow y \in B_r(x)$
$\therefore B_P(x) \subseteq B_r(x)$

Next, let $B_P(x), P > 0\ be\ d^* - open$

$\because d^*(x,y) \leq 1, \forall x,y \in X$

We take $0 < P < 1$

Let $B_r(x) be\ d - open, r = \frac{P}{1-P}$

T.P. $B_r(x) \subseteq B_P(x)$

Let $y \in B_r(x) \Longrightarrow d(x,y) < r$

$$\Longrightarrow \frac{d^*(x,y)}{1-d^*(x,y)} < \frac{P}{1-P} [d^*(x,y) = \frac{d(x,y)}{1+d(x,y)} \Longrightarrow d(x,y)$$
$$= \frac{d^*(x,y)}{1-d^*(x,y)}$$

$\Longrightarrow d^*(x,y) - Pd^*(x,y) < P - Pd^*(x,y)$

$\Longrightarrow d^*(x,y) < P$

$\Longrightarrow y \in B_P(x)$

$\therefore B_r(x) \subseteq B_P(x)$

$\therefore Every\ d - open\ is\ a\ d^* - open\ and\ conversely.$

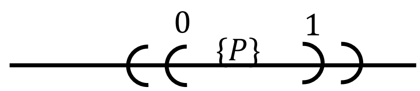
Definition: Let $(X.d)$ be a metric space, a point $P \in X$ is said to be a limit point of a set $A \subset C$, if every ball (nbd.) of P contains a point of P.

i.e. P is a limit point of A if $\forall B_r(P)$ then $[B_r(P) - \{P\}] \cap A \neq \phi$

Definition: The set of all points of a set $A \subset X$ is known as the derived set and is denoted by A'.

Example: let $(\mathbb{R}, d)$ be a metric space and $A = (0,1)$

Solution:



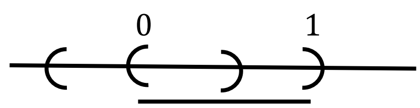
$\forall P \in (0,1) \Longrightarrow \forall B_\epsilon(P); \epsilon > 0$

$[B_\epsilon(P) - \{P\}] \cap (0,1) \neq \phi$

$\forall B_r(P), r > 0$

$[B_r(P) - \{P\}] \cap (0,1) \neq \phi$

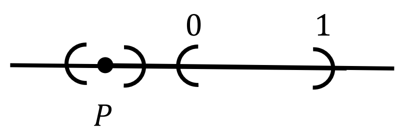
$\forall \epsilon > 0, \qquad \forall B_\epsilon(0)$



$[B_\epsilon(0) - \{0\}] \cap (0,1) \neq \phi$

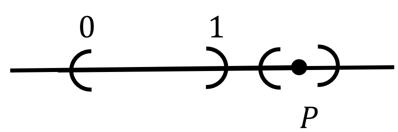
$[B_\epsilon(1) - \{1\}] \cap (0,1) \neq \phi, B_\epsilon(0)$

If $P < 0 \Longrightarrow \exists \epsilon > 0 \ s.t. B_\epsilon(P)$



$[B_\epsilon(P) - \{P\}] \cap (0,1) = \phi$

If $P > 1 \Longrightarrow \exists \epsilon > 0 \ s.t. B_\epsilon(P)$



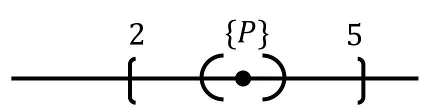
$[B_\epsilon(P) - \{P\}] \cap (0,1) = \phi$

$\therefore A' = [0,1]$

Example: let $(\mathbb{R}, d)$ be a metric space and $A = [2,5]$

Solution:

$\forall P \in [2,5] \Longrightarrow \forall B_\epsilon(P)$



$[B_\epsilon(P) - \{P\}] \cap [2,5] \neq \phi$

$r > 0, \qquad \forall B_r(P)$

$[B_r(P) - \{P\}] \cap [2,5] \neq \phi$

$\forall P < 2 \Longrightarrow \exists \epsilon > 0 \; s.t. B_\epsilon(P)$

$[B_\epsilon(P) - \{P\}] \cap [2,5] = \phi$

$\forall P > 5 \Longrightarrow \exists \epsilon > 0 \; s.t. B_\epsilon(P)$

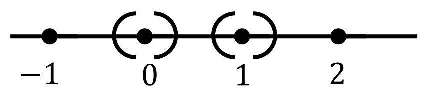
$[B_\epsilon(P) - \{P\}] \cap (0,1) = \phi$

$\therefore A' = [2,5]$

Example: let $(\mathbb{R}, d)$ be a metric space and $A = \mathbb{Z}$

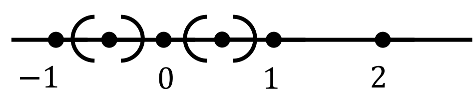
Solution:

$\forall P \in \mathbb{Z} \Longrightarrow \exists \epsilon > 0 \; s.t.$



$[B_\epsilon(P) - \{P\}] \cap \mathbb{Z} = \phi$

$\forall P \in \mathbb{R} - \mathbb{Z} \Longrightarrow \exists \epsilon > 0 \; s.t.$



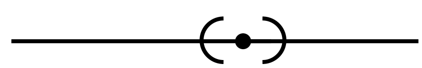
$[B_\epsilon(P) - \{P\}] \cap \mathbb{Z} = \phi$

$\therefore E' = \phi$

Example: let $(\mathbb{R}, d)$ be a metric space and $A = \mathbb{Q}$

Solution:

$\forall P \in \mathbb{Q}, \exists \epsilon > 0 \; s.t.$



$[B_\epsilon(P) - \{P\}] \cap \mathbb{Q} \neq \phi$

$\forall P \in \mathbb{Q} \, , \forall B_r(P) \; s.t.$

$[B_r(P) - \{P\}] \cap \mathbb{Q} \neq \phi$

$\forall P \in \mathbb{R} - \mathbb{Q} \, , \forall \epsilon > 0$

$[B_\epsilon(P) - \{P\}] \cap \mathbb{Q} \neq \phi$

$\forall P \in \mathbb{R} - \mathbb{Q} \, , \forall B_r(P)$

$[B_r(P) - \{P\}] \cap \mathbb{Q} \neq \phi$

$\therefore A' = \mathbb{R}$

Exercise: let $(\mathbb{R}, d)$ be a metric space find the derived set of $A = \{1,2, \dots ,10\}, B = \mathbb{N}, C = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}, D = [-4,2)$

Definition: A subset A of a metric space (X, d) is said to be closed if A contains all of its limit points.

i.e. $A \subseteq X$ is closed iff $A' \subseteq A$.

Example: let $(\mathbb{R}, d)$ be a metric space and $A = (0, ,1)$

$\because\ A' = [0,1]$ and $A' \nsubseteq A$

$\therefore A = (0,1)$ is not closed.

Example: let $(\mathbb{R}, d)$ be a metric space and $A = [2,7]$

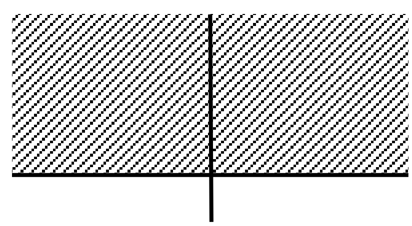
$\because\ A' = [2,7]$ and $A' \subseteq A$

$\therefore A = [2,7]$ is closed.

Example: let $(\mathbb{R}^2, d)$ be a metric space and

$$H = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$$

Solution:



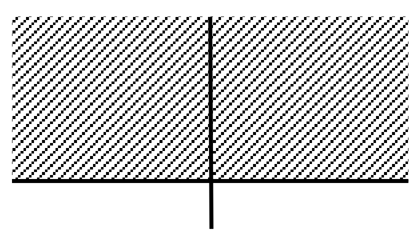
$H' = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ (check)

$\therefore H' \subseteq H$

$\therefore H$ is closed

Example: let $(\mathbb{R}^2, d)$ be a metric space and

$$K = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$



$K' = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$

$\therefore K' \nsubseteq K$

$\therefore K$ is not closed

Theorem 2.5: In a metric space a set E is closed if and only if its complement is open.

Proof: Suppose that E is closed set

T.P. E^c is open

Let $x \in E^c \Longrightarrow x \notin E$

$\because$ E is closed

$\therefore$ x is not a limit point of E

$\Longrightarrow \exists r > 0, s.t. B_r(x) \cap E = \phi$

$\Longrightarrow x \in B_r(x) \subseteq E^c$

$\Longrightarrow E^c$ is open

Suppose that E^c is open

T.P. E is closed

Let x be a limit point of E

$\therefore \forall B_r(x)$ s.t. $B_r(x) \cap E \neq \phi$

$\therefore B_r(x) \nsubseteq E^c$

$\because E^c$ is open

$\therefore x \notin E^c$

$\therefore x \in E$

$\therefore E$ is closed

Theorem 2.6: for any collection $\{E_i\}_{i\in I}$ of closed sets, then $\bigcap_{i\in I} E_i$ is closed.

Proof: let $\{E_i\}_{i\in I}$ is closed

T.P. $\bigcap_{i\in I} E_i$ is closed

$$\left(\bigcap_{i\in I} E_i\right)^c = \bigcup_{i\in I} E_i^c$$

$\because E_i^c$ is open set, $\forall i \in I$ [Theorem 2.5]

By theorem 2.2 we get

$\bigcup_{i\in I} E_i^c$ is open

Hence $\bigcap_{i\in I} E_i$ is closed [Theorem 2.5]

Theorem 2.7: for any finite collection $E_1, E_2, \dots, E_n$ of closed sets then $\bigcup_{i=1}^n E_i$ is closed.

Proof: suppose that $E_1, E_2, \dots, E_n$ are closed sets T.P. $\bigcup_{i=1}^n E_i$ is closed.

Since $(\bigcup_{i\in I}^n E_i)^c = \bigcap_{i\in I}^n E_i^c$

E_i^c is open set $i = 1, \dots, n$ [Theorem 2.5]

By theorem 2.3 we get

$$\bigcap_{i\in I}^n E_i^c \text{ is open}$$

Hence $\bigcup_{i\in I}^n E_i$ is closed.

Definition: In a metric space (X, d) the closure of a set E is denoted by $\bar{E}$ or $cl(E)$ which is defined by

$$\bar{E} = E \cup E'$$

Example: let $(\mathbb{R}, d)$ be a metric space, let $E = (0,1), A = \mathbb{Z}$.

Then $\bar{E} = E \cup E' = (0,1) \cup [0,1] = [0,1]$

$$\bar{A} = A \cup A' = \mathbb{Z} \cup \phi = \mathbb{Z}$$

Theorem 2.8: if (X, d) is a metric space and $E \subset X$ then:

a) $\bar{E}$ is closed,
b) $E = \bar{E}$ iff E is closed,
c) $\bar{E} \subseteq F$ for every closed set $F \subseteq X$ such that $E \subset F$.

Proof:

a) If $P \in X$ and $P \notin \bar{E}$
$\Longrightarrow P \notin E$ and $P \notin E'$
$\therefore (\bar{E})^c$ is open
$\therefore \bar{E}$ is closed [Theorem 2.5]
b) Suppose that $E = \bar{E}$
By (a) we get E is closed
Suppose that E is closed
$\Longrightarrow E' \subset E$
$\Longrightarrow E = E \cup E'$
$\Longrightarrow E = \bar{E}$
c) Suppose that $E \subseteq F$ and F is closed
T.P. $\bar{E} \subseteq F$
$\because F$ is closed
$\therefore F' \subseteq F$ and $E \subseteq F$
$\Longrightarrow E' \subseteq F$
$\Longrightarrow E \cup E' \subseteq F$
$\Longrightarrow \bar{E} \subseteq F$