Chapter - 3 - ①

Definition: Let $f: \mathbb{N} \longrightarrow \mathbb{R}$ be a function then $f(n) = p_n$, $\forall n \in \mathbb{N}$, is called a sequence of real numbers. which will be denoted by $\langle p_n \rangle$ or $\{p_n\}$.

$\langle p_n \rangle = p_1 , p_2 , \ldots , p_n , \ldots$

Ex:

$\langle \frac{1}{n} \rangle = 1, \frac{1}{2} , \frac{1}{3} , \ldots , \frac{1}{n} , \ldots$

$\langle (-1)^n \rangle = -1 , 1 , -1 , \ldots , (-1)^n , \ldots$

$\langle 3^n \rangle = 3 , 9 , 81 , \ldots , 3^n , \ldots$

$\langle \frac{1}{2} \rangle = \frac{1}{2} , \frac{1}{2} , \ldots , \frac{1}{2} , \ldots$

$\langle \frac{n}{n+1} \rangle = \frac{1}{2} , \frac{2}{3} , \frac{3}{4} , \ldots , \frac{n}{n+1} , \ldots$

Definition: Let (X,d) be a metric space and $p \in X$ a sequence $\{p_n\}$ is said to be convege to p if for each $\epsilon > 0$, there exists a positive integer $N \ni d(p_n,p) < \epsilon , \forall n \geq N$
if $\{p_n\}$ does not converge it is called (diverges)

Note: ① $\{p_n\}$ converge to p also mens p limit point of $\{p_n\}$ and we written $p_n \longrightarrow p$ or $\lim_{n \to \infty} p_n = p$.

② The set $\{P_1, P_2, \dots\}$ is called the range of $\{P_n\}$

Example: Let $(\mathbb{R}, d)$ be usuall metric space, show that the seq. $\{\frac{1}{n}\}$ converge to 0.

proof: Let $\epsilon > 0$,

T.p $\exists$ apositive integer $N \ni d(P_n, P) < \epsilon$ $\forall n \geq N$

i.e $\exists N \ni |\frac{1}{n} - 0| < \epsilon$

let N be the least positive integer $N > \frac{1}{\epsilon}$

if $n \geq N$ then $n > \frac{1}{\epsilon}$

$\Rightarrow \frac{1}{n} < \epsilon$

$\Rightarrow |\frac{1}{n}| < \epsilon$

$\Rightarrow |\frac{1}{n} - 0| < \epsilon$

$\Rightarrow d(P_n, P) < \epsilon$

$\therefore \{P_n\}$ is conv. to 0.

Theorem: Let $\{P_n\}$ be a sequence in a metric space (X, d) then $\{P_n\}$ converges to $p \in$ iff every nieghborhood of p contians all elements P_n expect a finite set.

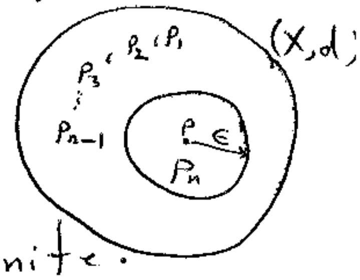
proof: $\Rightarrow$) Let $\{P_n\}$ conv. to $p \in X$ and B be aniegh. of p

Let $\epsilon > 0$ be radius of B

since $\{p_n\}$ is conv. to p.

$\therefore \exists$ positive integer $N \ni d(p_n, p) < \epsilon, \forall n \geq N$

Thus $\forall n \geq N$, $p_n \in B$



and $p_1, p_2, \dots, p_{n-1} \in B$

The set $\{p_1, p_2, \dots, p_{n-1}\}$ is finite.

$\Leftarrow)$

<u>Theorem</u>: Let $\{p_n\}$ be a seq. in a m.s. (X,d) if $p_1, p_2 \in X$ and $\{p_n\}$ conv. to p_1 and p_2 then $p_1 = p_2$.

<u>proof</u>: since $\{p_n\}$ conv. to p_1

$\Rightarrow \exists$ a positive integer $N_1 \ni d(p_n, p_1) < \frac{\epsilon}{2}$ $\forall n \geq N_1$

since $\{p_n\}$ conv. to p_2

$\Rightarrow \exists$ a positive integer $N_2 \ni d(p_n, p_2) < \frac{\epsilon}{2}$ $\forall n \geq N_2$

let $N = \max\{N_1, N_2\}$

$\therefore d(p_n, p_1) < \frac{\epsilon}{2}$ and $d(p_n, p_2) < \frac{\epsilon}{2}$ $\forall n \geq N$

$$d(p_1, p_2) \leq d(p_1, p_n) + d(p_n, p_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore d(p_1, p_2) < \epsilon \Rightarrow d(p_1, p_2) = 0 \Rightarrow p_1 = p_2$

Definition: Let (X, d) be a metric space and let q be a fixed point in X a subset E of X is called bounded if $\exists$ a positive real number $M \ni d(x, q) \leqslant M$, $\forall$ $x \in X$.

Theorem: Let $\{P_n\}$ be a seq. in M. S. (X, d) conv. to p then $\{P_n\}$ is bounded.

proof: Let $\epsilon = 1$

since $\{P_n\}$ conv. to p

$\Rightarrow$ Let $M = \max \{1, d(P_1, p), d(P_2, p), \dots, d(P_{n-1}, p)\}$

$\Rightarrow d(P_n, p) < M$ for $n = 1, 2, 3, \dots, n-1$

The range of $\{P_n\}$ is bounded

$\therefore \{P_n\}$ is bounded (by def. of bounded seq.)

Theorem: Let $\{P_n\}$ be a seq. in a m.s. (X, d) if $E \subseteq X$ and p is a limit point of E, then there is a seq. $\{P_n\}$ in $E \ni \lim_{n \to \infty} P_n = p$.

proof: $\forall$ a positive integer n

$\exists$ $B_{\frac{1}{n}}(p)$ s.t $(B_{\frac{1}{n}}(p) - \{p\}) \cap E \neq \emptyset$

(since p is limit point of E)

$\Rightarrow P_n \in E$ $\forall$ $n = 1, 2, \dots$

$\Rightarrow \{P_n\}$ is a seq. in E

$\therefore p_n \in B_{\frac{1}{n}}(p) \quad \forall n$

$\Rightarrow d(p_n, p) < \frac{1}{n} \quad \forall n$

let $\epsilon > 0$

$\Rightarrow \exists$ apositive integer $N \ni N > \frac{1}{\epsilon}$

if $n \geq N \Rightarrow n > \frac{1}{\epsilon} \Rightarrow \frac{1}{n} < \epsilon$

$\therefore d(p_n, p) < \epsilon \quad \forall n \geq N$

$\Rightarrow \lim_{n\to\infty} p_n = p$

Theorem: Suppose $\{s_n\}$, $\{t_n\}$ are two real seq. and $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$

then

① $\lim_{n\to\infty} (s_n + t_n) = s + t$

② $\lim_{n\to\infty} (K \cdot s_n) = K \cdot s$

③ $\lim_{n\to\infty} s_n \cdot t_n = s \cdot t$

④ $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ $\quad (s_n \neq 0, \ s \neq 0, \ \forall n = 1, 2, \dots)$

proof:

① Let $\epsilon > 0$ be given

since $\lim_{n\to\infty} s_n = s$

$\therefore \exists$ apositive integer $N_1 \ni |s_n - s| < \frac{\epsilon}{2}, \forall n \geq N$

since $\lim_{n\to\infty} t_n = t$

$\therefore \exists$ apositive integer $N_2 \ni |t_n - t| < \frac{\epsilon}{2}, \forall n \geq$

let $N = \max\{N_1, N_2\}$

$$|(s_n + t_n) - (s+t)| = |(s_n - s) + (t_n - t)|$$
$$\leqslant |s_n - s| + |t_n - t|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore |(s_n + t_n) - (s+t)| < \epsilon \quad \forall n \geqslant N$

$\therefore \lim_{n \to \infty} (s_n + t_n) = s + t$

② $\lim_{n \to \infty} k s_n = k s$

let $\epsilon > 0$ be given since $\lim_{n \to \infty} s_n = s$

$\Rightarrow \exists$ positive integer N

$\ni |s_n - s| < \frac{\epsilon}{|k|} \quad \forall n \geqslant N, \; k \neq 0$

$$|k s_n - k s| = |k(s_n - s)|$$
$$= |k| \cdot |s_n - s|$$
$$= |k| \cdot \frac{\epsilon}{|k|}$$
$$= \epsilon$$

$\therefore |k s_n - k s| < \epsilon$

$\Rightarrow \lim_{n \to \infty} k s_n = k s$

<u>Definition</u>: Let (X,d) be a metric space a sequence $\{P_n\}$ is Cauchy sequence if for each $\epsilon > 0$, $\exists$ a positive integer $N \ni$ $d(P_n, P_m) < \epsilon$, $\forall\, n \geq N,\ m \geq N$.

<u>Example</u>: Let $(\mathbb{R}, d)$ be the usuall metric space show that a sequence $\{\frac{1}{n}\}$ is Cauchy'

<u>Solution</u>: Let $\epsilon > 0$ be given

let N be the smallest positive integer

$\ni \quad N > \frac{2}{\epsilon} \quad \forall\, n \geq N,\ m \geq N.$

$\because\ n \geq N$

$\Rightarrow \frac{1}{n} \leq \frac{1}{N} \quad \Rightarrow \left|\frac{1}{n}\right| \leq \frac{1}{N}$

$\because\ N > \frac{2}{\epsilon} \quad \Rightarrow \frac{1}{N} < \frac{\epsilon}{2}$

$\therefore\ \left|\frac{1}{n}\right| < \frac{\epsilon}{2}$

By the same way

$\left|\frac{1}{m}\right| < \frac{\epsilon}{2}$

$$\therefore\ d(P_n, P_m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore\ d(P_n, P_m) < \epsilon$

$\therefore\ \{\frac{1}{n}\}$ is Cauchy seq.

Theorem: In any metric space every convergent sequence is a Cauchy sequence.

proof: let (X, d) be a metric space

let $\{X_n\}$ is convergent seq. in X

$\ni$ $\{X_n\}$ convergent to point x_0

T.P $\{X_n\}$ is a cauchy seq.

let $\epsilon > 0$ be given

since $\{X_n\}$ convergent to point $x_0 \in X$

$\therefore \exists$ positive integer $N \ni d(X_n, x_0) < \frac{\epsilon}{2}, \forall n \geq N$

$\forall \; n \geq N \; , \; m \geq N$

$$d(X_n, X_m) \leq d(X_n, x_0) + d(x_0, X_m)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore d(X_n, X_m) < \epsilon$

$\therefore \{X_n\}$ is a cauchy seq.

Remark: The converse is not necessary true ((that is cauchy seq. is not necessary to be convergent)).

proof: let $(\mathbb{R} - \{0\}, d)$ be usuall metric space.

The seq. $\{\frac{1}{n}\}$ is cauchy in a metric sp. $(\mathbb{R} - \{0\}, d)$ but not convergent seq.

becase $\{\frac{1}{n}\}$ convergent to 0

and $0 \notin (\mathbb{R}-\{0\}, d)$

That $\{\frac{1}{n}\}$ diverges.

Remark: A bounded sequence is not necessary to be a convergent seq.

Example: let $(\mathbb{R}, d)$ be a usuall metric sp. the seq $\{(-1)^n\}$ is bounded but doesn't convergent, because $\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}$

$= \{-1, 1\}$

Theorem:

1) If $p > 0$ then $\lim_{n \to \infty} \frac{1}{n^p} = 0$

2) If $p > 0$ then $\lim_{n \to \infty} \sqrt[n]{p} = 1$

3) $\lim_{n \to \infty} \sqrt[n]{n} = 1$

4) If $p > 0$ and α is real then $\frac{n^\alpha}{(1+p)^n} = 0$

5) If $|x| < 1$ then $\lim_{n \to \infty} x^n = 0$

proof: ① T.P $\lim_{n \to \infty} \frac{1}{n^p} = 0$

that is to prove $\exists$ positive integer N

$\ni \left| \frac{1}{n^p} - 0 \right| < \epsilon$, $\forall n \geq N$

let N be the smallest positive integers

$\ni \quad N > \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}} \quad , \quad \forall \; n \geqslant N$

since $\quad N > \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}$

$\therefore \; n \geqslant N \quad \Rightarrow \quad n > \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}$

$\Rightarrow \quad n^p > \frac{1}{\epsilon}$

$\Rightarrow \quad \frac{1}{n^p} < \epsilon \qquad \Rightarrow \left|\frac{1}{n^p}\right| < \epsilon$

$\therefore \; \left|\frac{1}{n^p} - 0\right| < \epsilon$

Thus $\lim\limits_{n \to \infty} \frac{1}{n^p} = 0$

(2) proof: if $p = 1$ then $\lim\limits_{n \to \infty} \sqrt[n]{1} = 1$

if $p > 1$

let $\quad X_n = \sqrt[n]{p} - 1$

$\Rightarrow \quad \sqrt[n]{p} = x_n + 1$

$$\begin{aligned}
\Rightarrow \quad p &= (x_n + 1)^n \\
&= (1 + X_n)^n \\
&= 1 + n X_n + \frac{n(n-1) X_n^2}{2!} + \cdots + \frac{X^n}{n!}
\end{aligned}$$

$\therefore \; p \geqslant 1 + n X_n$

$\Rightarrow \quad X_n \leqslant \frac{p-1}{n}$

$$0 < X_n < (p-1)\cdot\frac{1}{n}$$

as $n \longrightarrow \infty$, $X_n \longrightarrow 0$

$$\lim_{n\to\infty} X_n = 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\sqrt[n]{p} - 1\right) = 0$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{p} - 1 = 0$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{p} = 1$$

If $0 < p < 1$

$$p = \frac{1}{q} \quad \Rightarrow \quad q > 1$$

$$\lim_{n\to\infty} \sqrt[n]{p} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{q}}$$

$$= \frac{\lim_{n\to\infty} \sqrt[n]{1}}{\lim_{n\to\infty} \sqrt[n]{q}} = \frac{1}{1} = 1$$

Sequence and Series of functions

Suppose $F(S) = \{ f : f : S \longrightarrow \mathbb{R} \}$ is a set of real valued functions defined on a subset $S \subseteq \mathbb{R}$.

Let $\{ f_n \}_{n=1}^{\infty}$ be a sequence in $F(S)$ such that $\{ f_n(x) \}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$.

If $\{ f_n(x) \}_{n=1}^{\infty}$ is a convergent on $S \subseteq \mathbb{R}$, thus $f(x) = \lim_{n \to \infty} f_n(x)$ $\forall x \in S$.

Definition (convergence in $\mathbb{R}$)

$\forall \epsilon > 0 \ \exists k \in \mathbb{Z}_+ \ni | f_n(x) - f(x) | < \epsilon \quad \forall n > k$ and for some x.

k depending on ϵ and x or only on ϵ.

Definition (Point wise convergence)

$$\forall \epsilon > 0, \exists k \in \mathbb{Z}_+ \ni | f_n(x) - f(x) | < \epsilon$$

$$\forall n > k(\epsilon, x)$$

(i.e k depending on ϵ, x) $\quad x \in S \subseteq \mathbb{R}$.

Example

Let $\{ f_n(x) = \frac{x}{n} \}_{n=1}^{\infty}$ be a sequence of functions defined on $\mathbb{R} \ni f_n(x) = \frac{x}{n} \quad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$

by archi-properety

$\frac{x}{k} < \epsilon \qquad k \in \mathbb{Z}_+, \epsilon > 0$

$n > k \Rightarrow \frac{1}{n} < \frac{1}{k} \Rightarrow \frac{x}{n} < \frac{x}{k} < \epsilon$

$\therefore \left|\frac{x}{n}\right| < \left|\frac{x}{k}\right| < \epsilon \quad \forall n > k$

$\therefore \frac{x}{n} \to 0 \Rightarrow f_n(x) \to 0$ as $n \to \infty$

if k does not depending on x

$\Rightarrow k$ is fixed

$\Rightarrow \frac{x}{k} < \epsilon \Rightarrow x < k\epsilon \Rightarrow x$ is bounded (C!)

but $x \in \mathbb{R}$ is unbounded.

$\therefore$ K depending on x and ϵ.

<u>Definition</u> (Uniformily convergence)

$\forall \epsilon > 0 \; \exists k \in \mathbb{Z}_+ \ni |f_n(x) - f(x)| < \epsilon \quad \forall n > k(\epsilon)$.

(i.e K depending only on ϵ).

<u>Example</u>

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence defined on $(0, b)$

$f_n(x) = \frac{x}{n}$, $x \in (0, b)$, $n \in \mathbb{N}$

Let $\epsilon > 0$, $\exists k \in \mathbb{N} \ni$

$\frac{x}{k} < \epsilon$ (by archi. property)

$n > k \Rightarrow \frac{1}{n} < \frac{1}{k} \Rightarrow \frac{x}{n} < \frac{x}{k} < \epsilon$, $k \in \mathbb{Z}_+$

$\because x \in (0, b) \Rightarrow x < b$, $k \in \mathbb{Z}_+$

$\Rightarrow \frac{x}{n} < \frac{b}{n} < \epsilon \Rightarrow \frac{b}{k} < \epsilon \Rightarrow K$ depending on ϵ only

since b constant.

$\Rightarrow f_n(x) = \frac{x}{n} \to f(x) = 0$ uniformly conv.

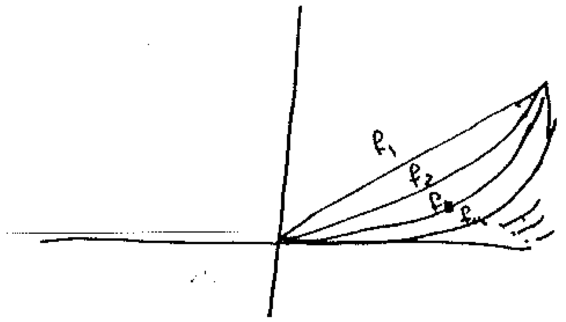
Remark

Every uniformly continuous Sequence of functions is pointwise continuous Sequence of functions, but the converse is not true as following example.

Example

Let $\{f_n\}_{n=1}$ be a sequence defined on $[0,1]$ such that

$f_n(x) = x^n \quad , \quad x \in [0,1] \quad , \quad n \in \mathbb{N}.$



(i) $\forall x, \ 0 < x < 1$

x^n is decreasing sequence an 0's is a lower bounded

Let $\epsilon > 0 \ \exists k \in \mathbb{Z}_+ \ni$

$$k \log x < \log \epsilon \Rightarrow \log x^k < \log \epsilon \Rightarrow$$

$$x^k < \epsilon \quad \forall n > k \quad , 0 < x < 1$$

$$\therefore \ x^k \to 0 . \quad 0 < x < 1$$

(ii) $x = 0 \Rightarrow x^n \to 0 \quad \forall n \in \mathbb{N}.$

$x = 1 \Rightarrow x^n \to 1 \quad \forall n \in \mathbb{N}.$

$$\therefore f_n \to f \ni f_n(x) \to f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

∴ to show that $f_n(x) = x^n$ is not uniformly conv.

or the equation is (Is there exist k depending only on ϵ)

$$|x^n| < \epsilon \qquad \forall n > k, \ \forall x \in (0,1)$$

Let $x_n = 2^{-\frac{1}{n}} \Rightarrow x_n = \frac{1}{2^{\frac{1}{n}}} \Rightarrow x_n^n = \frac{1}{2}$

and let $\epsilon = \frac{1}{4} \Rightarrow |x_n^n| = |\frac{1}{2}| < \frac{1}{4}$ (C!)

∴ $f_n(x)$ doesnot convergent uniformly on $x \in (0,1)$

$\Rightarrow$ $f_n(x)$ does not convergent uniformly on $x \in [0,1]$.

H.W

check that $\{f_n(x) = x^n\}_{n=1}^{\infty}$ is uniformly ~~continuous~~ on $x \in [0,a]$, $0<a<1$.

Example

Let $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$

(i) $x = 0 \Rightarrow f_n(x) = 0 = f(x)$

(ii) $x \neq 0 \Rightarrow$ For $\epsilon > 0$

$$\left|\frac{xn}{1+n^2x^2}\right| < \left|\frac{nx}{n^2x^2}\right| = \left|\frac{1}{nx}\right| < \left|\frac{1}{kx}\right| < \epsilon$$

∴ $\left|\dfrac{nx}{1+n^2x^2}\right| < \epsilon \quad \forall n > k$

∴ $f_n(x) \to 0$, $x \in \mathbb{R}$

(iii) $x_n = \frac{1}{n}$, $n \in \mathbb{N}$

$\because f_n(\frac{1}{n}) = \frac{1}{2}$

Let $\in = \frac{1}{4}$

$|f_n(\frac{1}{n}) - f(\frac{1}{n})| = |\frac{1}{2} - 0| < \frac{1}{4}$ (c!)

$\therefore f_n$ does not converget uniformly.

(iv) if $f_n \to f = 0$ on (a, ∞)

$|f_n(x)| = |\frac{nx}{1+n^2x^2}| < |\frac{nx}{n^2x^2}| = |\frac{1}{nx}| < \frac{1}{na} < \frac{1}{ka} < \in$

$\therefore$ k depending only on $\in$

$\therefore f_n \to f$ uniformly conv.

Theorem

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of bounded functions converges uniformly to a function f. Then f is bounded.

Proof

Suppose $f_n \to f$ as $n \to \infty$ uniformly

$\Rightarrow \forall \in > 0 \ \exists k \in \mathbb{Z}_+ \ni |f_n(x) - f(x)| < \in \quad \forall x \in S \subseteq \mathbb{R}$ and $n > k(\in)$.

Take $\in = 1 \Rightarrow |f_n(x) - f(x)| < 1$

Since $|f_{k+1}(x)| \leq M$ (f_{k+1} bounded), M positive number.

$\therefore |f(x)| = |f(x) - f_{k+1}(x) + f_{k+1}(x)| \leq |f(x) - f_{k+1}(x)| + |f_{k+1}(x)|$

$\therefore \quad |f(x)| \leq 1+M$

$\therefore$ $f(x)$ is bounded $\forall x \in S \subseteq \mathbb{R}$.

Theorem

Let $\{f_n(x)\}$ be a sequence of continuous functions on $S \subseteq \mathbb{R}$ and converges uniformly to a function $f(x)$. Then $f(x)$ is continuous on $S \subseteq \mathbb{R}$.

proof

By definition of continuous in a metric space.

$x_n \to x$ in $D_f \implies f(x_n) \to f(x)$ in R_f.

Let $x_n \to x$ as $n \to \infty$

$$|f(x_n) - f(x)| = |f(x_n) - f_n(x_n) + f_n(x_n) - f_n(x) + f_n(x) - f(x)|$$
$$\leq |f(x_n) - f_n(x_n)| + |f_n(x_n) - f_n(x_n)| + |f_n(x) - f(x)|$$

since $f_n \to f$ uniformly conv.

$\therefore \forall \epsilon > 0 \quad \exists k \in \mathbb{Z}_+ \ni \quad |f_n(x) - f(x)| < \frac{\epsilon}{3}$

$\forall x \in S \subseteq \mathbb{R}, \; n > \ell, \; \ell \in \mathbb{Z}_+$

$\because$ and $|f_n(x) - f(x)| < \frac{\epsilon}{3}$, $\forall \epsilon > 0$
$n > k$, $k \in \mathbb{Z}_+$

$\therefore$ $|f(x_n) - f(x)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$
$\forall n > \max\{k, l\}$

$\therefore$ $f(x_n) \to f(x)$.

$\therefore$ f continuous function on $S \subseteq \mathbb{R}$.