Chapter -4-

<u>Definition</u>: Let $\{a_n\}$ be a sequance of real number. An infinite series $\sum_{n=1}^{\infty} a_n$ is defined to be the sequance $\{S_n\}$ where $S_n = \sum_{k=1}^{n} a_k$
The number S_n is called the partial sum of the series.

<u>Definition</u>: The series $\sum_{n=1}^{\infty} a_n$ is said to be converges if the sequance $\{S_n\}$ converges to s.

if the sequance $\{S_n\}$ diverges then infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Remark</u>: If the series $\sum a_n$ converges to s we write $\sum_{n=1}^{\infty} a_n = s$.
(s is called the sum of series) or write $\lim_{n\to\infty} S_n = s$

<u>Example</u>: $\sum_{n=1}^{\infty} a r^{n-1}$, where $a \neq 0$ is an infinite series called Geometric series.

<u>Solution</u>: $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

if $r=1$ $\therefore$ $S_n = a + a + \cdots + a$
$\Rightarrow S_n = na$ $\therefore \{S_n\}$ diverges $\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ div.

if $r \neq 1$

$$\therefore \quad S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$-r S_n = -ar - ar^2 \quad \cdots \quad -ar^{n-1} - ar^n$$

$$S_n - r S_n = a - ar^n$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

if $|r| < 1 \Rightarrow \{r^n\} \longrightarrow 0$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \lim_{n\to\infty} \frac{a}{1-r}$$

$$\therefore \quad \{S_n\} \longrightarrow \frac{a}{1-r} \quad \text{conv.}$$

$$\therefore \quad \sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r} \quad \text{conv.}$$

if $|r| > 1 \Rightarrow \{r^n\}$ diverges

$$\Rightarrow \{S_n\} = \left\{\frac{a(1-r^n)}{1-r}\right\} \text{ diverges}$$

$$\therefore \quad \sum ar^{n-1} = \begin{cases} \text{converges} & \text{where } -1 < r < 1 \\ \text{diverges} & \text{where } r \geq 1 \text{ or } r \leq -1 \end{cases}$$

Example : $\sum \frac{1}{n}$ is an infinite series called the harmonic series.

Solution : $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+n}$$

$$S_{2n} - S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$\frac{1}{n+1} \geqslant \frac{1}{n+n}$$

$$n+1 \leqslant n+n$$

$$\geqslant \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2\cancel{n}} \cdot \cancel{n} = \frac{1}{2}$$

$\therefore |S_{2n} - S_n| \geqslant \frac{1}{2}$

$\therefore \{S_n\}$ is not cauchy sequance

$\therefore \{S_n\}$ diverges.

Theorem: If $\sum_{n=1}^{\infty} a_n$ conve. to A and $\sum_{n=1}^{\infty} b_n$ conve. to B then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $A+B$

proof: If $S_n = a_1 + a_2 + \cdots + a_n$

and $t_n = b_1 + b_2 + \cdots + b_n$

then $\lim_{n\to\infty} S_n = A$, $\lim_{n\to\infty} t_n = B$

i.e $\langle S_n \rangle \longrightarrow A$ and $\langle t_n \rangle \longrightarrow B$

let
$$\begin{aligned} v_n &= \sum_{i=1}^{n} (a_i + b_i) \\ &= (a_1+b_1) + (a_2+b_2) + \cdots + (a_n+b_n) \\ &= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) \\ &= S_n + t_n \end{aligned}$$

$\therefore \; V_n = S_n + t_n$

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} S_n + \lim_{n\to\infty} t_n$$

$$= A + B$$

$$\therefore \langle V_n \rangle \longrightarrow A+B$$

Thus $\sum_{n=1}^{\infty} a_n + b_n = A+B$.

<u>Theorem</u> : If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges

<u>Proof</u>: Suppose $\sum a_n + b_n$ converges

By theo.

$$\sum (a_n+b_n) - \sum a_n = \sum_{n=1}^{\infty} b_n \text{ converges Cl}$$

$\therefore \; \sum (a_n + b_n)$ diverges.

<u>Theorem</u>: $\sum a_n$ converges iff for any $\epsilon > 0$

There is an integer $N \ni \left| \sum_{k=n}^{m} a_k \right| < \epsilon$

if $m \geq N$, $n \geq N$.

<u>proof</u>: since $\sum a_n$ converges

$\therefore \{S_n\}$ is conv.

$\therefore \{S_n\}$ is Cauchy

$\therefore \exists$ a positive integer $N \ni |S_n - s| < \frac{\epsilon}{2}, \forall n \geq N$

if $m \geq N$, $n \geq N$

Then $|S_n - s| < \frac{\epsilon}{2}$ and $|S_m - s| < \frac{\epsilon}{2}$

$$|S_m - S_n| = |S_m - s + s - S_n|$$

$$\leq |S_m - s| + |S_n - s|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore \left| \sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k \right| < \epsilon$ if $m \geq n$

$\therefore \left| \sum_{k=n}^{m} a_k \right| < \epsilon$.

<u>Theorem</u>: if $\sum a_n$ converges then $\lim_{n \to \infty} a_n = 0$

<u>proof</u>: since $\sum a_n$ converges, $\forall \epsilon > 0$

$\exists$ positive integer $\ni \left| \sum_{k=n}^{m} a_k \right| < \epsilon$

$\forall n \geq N, m \geq N$

By theorem

$$\left| \underbrace{a_{n+1}}_{m} \right| = \left| \sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k \right| < \epsilon$$

$$\therefore \ |a_{n+1}| < \epsilon$$

This means $\lim_{n \to \infty} a_n = 0$.

Remark: The converse is not necessary to be true.

Example : Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n}$

Solution : $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$

But the series $\sum \frac{1}{n}$ is divergent series .

Definition : If $\sum_{k=1}^{\infty} |a_k|$ converges we say the series $\sum_{k=1}^{\infty} a_k$ converges absolutely, if $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges Then we say the series $\sum_{k=1}^{\infty} a_k$ converges condition .

<u>Theorem</u>: If $\sum_{k=1}^{\infty} |a_k|$ converges then

$\sum_{k=1}^{\infty} a_k$ converges.

<u>proof</u>: let $S_n = \sum_{k=1}^{n} a_k$, $A_n = \sum_{k=1}^{n} |a_k|$

since $\sum_{k=1}^{\infty} |a_k|$ converges

$\therefore$ $\{A_n\}$ converges

$\therefore$ $\{A_n\}$ is Cauchy seq.

Given $\epsilon > 0$ $\exists$ a positive integer $N \ni$

$$|A_m - A_n| < \epsilon \quad , \forall n \geq N , m \geq N$$

$$\left| \sum_{k=1}^{m} |a_k| - \sum_{k=1}^{n} |a_k| \right| < \epsilon$$

$$\sum_{k=n+1}^{m} |a_k| < \epsilon$$

since $\left| \sum_{k=n+1}^{m} a_k \right| \leq \sum_{k=n+1}^{m} |a_k|$

$\therefore$ $\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon$

$\therefore$ $|S_m - S_n| < \epsilon$

$\therefore$ $\{S_n\}$ is Cauchy sequance

$\therefore$ $\{S_n\}$ converges [every cauchy seq. of real number is converges in the field of real num.

$\therefore$ $\sum_{k=1}^{\infty} a_k$ converges.

Infinite Series of Functions

Let $\sum_{n=1}^{\infty} f_n$ be infinite series of function defined on Domain $D \subseteq \mathbb{R}$, and it's n-term defined as

$S_1(x) = f_1(x) \quad , x \in D$

$S_2(x) = f_1(x) + f_2(x) \quad , x \in D$

$\vdots$

$S_n(x) = f_1(x) + f_2(x) + \dots + f_n(x) \quad , x \in D$

Notes :

we said $\sum f_n$ is convergent at $x \in D$ if it's n-term sequence $\{S_n(x)\}_{n=1}^{\infty}$ is convergent

(i.e $S_n \longrightarrow S \Rightarrow \sum f_n = S$ or

$\sum f_n(x) = S(x) \; , \; x \in D$)

<u>Example</u>

show that $\sum_{n=1}^{\infty} x^{n-1}$ is convergent at $|x| < 1$ and divergent otherwise.

<u>Proof</u>

$\sum_{n=1}^{\infty} x^{n-1}$ is a geometric series and

$$S_k(x) = \sum_{n=1}^{k} x^{n-1} = \frac{1-x^k}{1-x}, \quad x \neq 1$$

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$\therefore$ $\{S_k(x)\}$ is convergent at $|x|<1$ and

$$S_n(x) \longrightarrow \frac{1}{1-x}$$

also

$\{S_k(x)\}$ is divergent at $|x| \geq 1$.

<u>Note</u>

The <u>importanc</u> of uniformly continuous of infinte series is the following theorem.

<u>Theorem</u>

Let $\sum_{n=1}^{\infty} f_n$ infinite series, where f_n are <u>continuous</u> on D, and $\sum_{n=1}^{\infty} f_n$ is uniformly continuous to f, then f is <u>continuous</u> function.

<u>Proof</u> (H.W)

Hint (Use above theorem of uniformly convergent of continuous Sequence)

Power Series

$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

where $a_n \in \mathbb{R}$, we called this type of series is a power series.

Ingeneral

$a_0 + a_1 (x-a) + a_2 (x-a)^2 + \dots + a_n (x-a)^n + \dots$

Let $y = x - a$, we get

$a_0 + a_1 y + a_2 y^2 + \dots + a_n y^n + \dots$ = first type, so we will study first type only which is at $a = 0$.

Note

at $x = 0$, $\sum a_n x^n$ convergent to a_0, so at least one convergence point of the series.

Example

$$\sum_{n=1}^{\infty} (n-1)!\, x^{n-1} = 1 + x + 2!x^2 + \dots +$$

Let $x \neq 0$,

if the series is convergent then $(n-1)!\, x^{n-1} \to 0$

$\Rightarrow \{(n-1)!\, x^{n-1}\}_{n=1}^{\infty}$ is bounded

$\Rightarrow |(n-1)!\, x^{n-1}| < M$, for only $x \in D$.

$\Rightarrow |(n-1)!| < \dfrac{M}{x^{n-1}}$, $x \neq 0$ C!

But $(n-1)!$ is not bounded as $n = 1, 2, \dots$

$\Rightarrow (n-1)!\, x^{n-1} \not\to 0 \Rightarrow \sum_{n=1}^{\infty} (n-1)!\, x^{n-1}$ is not conv.

Example

$$\sum \frac{x^{n-1}}{r^{n-1}} = 1 + \frac{x}{r} + \frac{x^2}{r^2} + \dots + \frac{x^n}{r^n} + \dots$$

$r > 0$

Geometric series convergent only at $\left|\frac{x}{r}\right| < 1$

$\Rightarrow |x| < |r|$

if $r > 0 \Rightarrow$ $\cancel{|x| < r}$ conv.

$|x| \geq r$ div.

Interval of convergence of a power series

Theorem

Let $\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$ is a ~~power~~ series convergent to $x_0 \neq 0$, then its convergent at all x such that $|x| < |x_0|$.

Proof

Let $\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$ is a power series and suppose that converges to $x_0 \neq 0$, if

$|x| < |x_0| \Rightarrow \sum_{n=1}^{\infty} |a_{n-1} x^{n-1}| = \sum_{n=1}^{\infty} |a_{n-1} x_0^{n-1}| \left|\frac{x}{x_0}\right|^{n-1}$

Since $\sum_{n=1}^{\infty} a_{n-1} x_0^{n-1}$ converges $\Rightarrow a_{n-1} x_0^{n-1} \longrightarrow 0$ as $n \to \infty$

$\therefore \exists M > 0 \ni |a_{n-1} x_0^{n-1}| \leq M, \forall n \in \mathbb{N}$

$$\Rightarrow \sum_{n=1}^{\infty} |a_{n-1} x^{n-1}| \leq M \sum_{n=1}^{\infty} \left|\frac{x}{x_0}\right|^{n-1}.$$

But $\sum_{n=1}^{\infty} \left|\frac{x}{x_0}\right|^{n-1}$ is a geometric series

and $\left|\frac{x}{x_0}\right| < 1$

$\therefore \sum \left|\frac{x}{x_0}\right|^{n-1}$ converges

$\Rightarrow \sum |a_{n-1} x^{n-1}|$ is also conv. for $|x| < |x_0|$