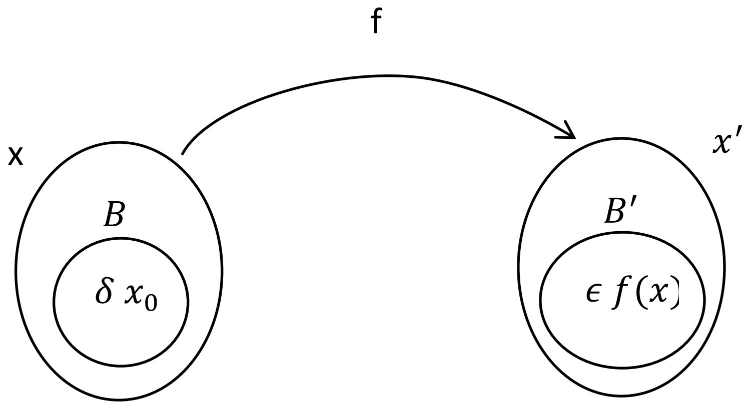
Chapter Five

The Continuity

Def.:

Let (x, d) and (x', d') be metric spaces and let $f: x \to x'$ be a function. f is said to be continuous at $x_0 \epsilon X$, if $\forall \epsilon > 0, \exists \delta = \delta(x_0, \epsilon)$ such that for any $x \epsilon X$ if $d(x, x_0) < \delta$ then $d'(f(x), f(x_0) < \epsilon$.

i.e. $f: x \to x'$ is continuous at $x_0 \; \epsilon X$, if for any ball (neigh.) in x' with center $f(x_0)$ and radius $\epsilon, B'_\epsilon\big(f(x_0)\big)$, there exists a ball $B_\delta(x_0) \; in \; x$ with center x_0 and radius δ such that $f(B) \subseteq B'$.



If f is continuous at each $x_0 \epsilon X$, then we say that f is continuous on x' .

Theorem 5.1:

Let $f: x \to x'$ be a function then f is continuous at x_0 iff for any open set v in X' with $f(x_0) \epsilon v, f^{-1}(v) is \; open \; in \; X$ where $f^{-1}(v) = \{x \epsilon X: f(x) \epsilon V\}$

Proof: (⇒) suppose that f is cont.

Let V be an open set in x' such that $f(x_0)\epsilon V$.

T.P. $f^{-1}(v) is\ open\ in\ X$.

∵ $f(x_0)\epsilon v$, V is open

⇒ ∃ a ball (neigh.) $B'\big(f(x_0)\big) \subseteq V$ (by def. of open set)

∵ f is cont.

∴ ∃ a ball (neigh.) $B\ in\ X$ such that $x_0 \epsilon B\ and\ f(B) \subseteq B'$ [by def. of cont.], $B' \subseteq V$

∴ $B \subseteq f^{-1}(v)$

(⇐) Suppose that every open V in $X', f^{-1}(v)$ is open in X

T.P. f is cont.

Let $x_0 \epsilon X, f(x_0) \epsilon X'$, and $B'_\epsilon\big(f(x_0)\big)$be a ball (neigh.) in X' with center $f(x_0)$ and radius G.

T.P. ∃ a ball $B(x_0)$ s.t. $f(B) \subseteq B'$.

∵ B' is an open set in $X', f(x_0)\epsilon B'$.

∴ $by\ the\ assumption\ f^{-1}(B')$ is open in X.

Clearly $x_0 \epsilon f^{-1}(B')$ [since $f(x_0)\epsilon B'$]

∴ ∃ a ball B in X s.t. $B(x_0) \subseteq f^{-1}(B')$

∴ $f\big(B(x_0)\big) \subseteq B'\big(f(x_0)\big)$

Theorem 5.3:

Let (x,d) and (x',d') be two metric spaces, $f: x \to x'$ is a mapping, f is cont. at $x_0 \epsilon X$, iff for every sequence $< x_n >$ converges to $x_0 \epsilon X$ the sequence $< f(x_n) >$ converges to $f(x_0)$.

<u>Proof:</u> $(\Rightarrow)$ suppose that f is cont. at x_0 and let $< x_n >$ be a sequence in X. that conv. to $x_0 \epsilon X$.

T.P. $< f(x_n) >$ converges to $f(x_0)$

$\because f: x \to x'_{f(x_0)}$

Let v be any open set in x' s.t. $f(x_0) \epsilon v$

$\because f$ is cont. at x_0

$\therefore f^{-1}(v)$ is open in x [th. 5.1]

$x_0 \epsilon f^{-1}(v) \Rightarrow f(x_0) \epsilon v \qquad [\because f(x_0) \epsilon v \Rightarrow x_0 \epsilon f^{-1}(v)]$

$\because x_n \to x_0$

$\therefore f^{-1}(v)$ contains most of the terms of the seq. $< x_n >$.

i.e. v contains most of the terms of the seq. $< f(x_n) >$.

(كل open set تحتوي نقطة التقارب تحتوي معظم حدود المتتابعة)

(f^{-1} هي open set وتحتوي x_0 والـ x_0 نقطة تقارب اذن $f^{-1}(x_0)$ تحتوي معظم حدود المتتابعة).

$\therefore f(x_n) \to f(x_0)$

Suppose that every seq. $< x_n >$ conve. To $x_0 \epsilon X$ the seq.

$< f(x_n) >$ conv. to $f(x_0)$.

T.P. f is cont.

Assume that f is not cont.

$\therefore \exists \epsilon > 0$ s.t. $\forall n \epsilon N, \quad \delta = \frac{1}{n}, \quad \exists x_n \epsilon X$

s.t. if $d(x_n, x_0) < \frac{1}{n}$ then $d'\big(f(x_n), f(x_0)\big) \geq \epsilon$

i.e. $\exists$ sequence $< x_n >$ in x s.t. $x_n \to x_0 \epsilon X$

$$(\therefore \epsilon > 0\ , \exists k \epsilon z^+\ s.t. \frac{1}{k} < \epsilon \Rightarrow d(x_n, x_0) < \frac{1}{n} < \frac{1}{k} < \epsilon, \therefore n > k\ )$$

But $f(x_n) \nrightarrow f(x_0)$ contradiction (with assup.)

$\therefore f$ is cont. at x_0

Example 1: let $f: \mathbb{R} \to \mathbb{R}, f(x) = c, c\epsilon\mathbb{R}, \forall x \epsilon \mathbb{R}$ f is cont.

Sol: let $x_0 \epsilon \mathbb{R}$

T.P. f is cont. at x_0

Let $< x_n >$ be a seq. in $\mathbb{R}$ s.t. $x_n \to x_0$

T.P. $f(x_n) \to f(x_0)$

$c'' \to c'' = f(x_0)$

$\therefore f\ is\ cont.$

Example 2: let $f: \mathbb{R} \to \mathbb{R},\ f(x) = x, \forall x \epsilon \mathbb{R}$

Sol: let $< x_n >$ in $\mathbb{R}$ s.t. $< x_n > \to x_0, x_0 \epsilon \mathbb{R}$

$f(x_n) = x_n \to x_0 = f(x_0)$

$\therefore f(x_n) \to f(x_0)$

$\therefore f$ is cont.

Example 3: let $f: \mathbb{R}^+ \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}, \forall x \epsilon \mathbb{R}^+$ then f is cont.

Sol: let $x_0 \epsilon \mathbb{R}^+$, and let $\epsilon > 0$

T.P. $\exists \delta(\epsilon, x_0)$ s.t. if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$

$$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x - x_0}{x x_0}\right| = \frac{|x - x_0|}{x x_0}$$

$$\frac{|x - x_0|}{x x_0} < \frac{|x - x_0|}{x_0} < \epsilon$$

$|x - x_0| < x_0 \epsilon$, choose $\delta = min.\{1, x_0 \epsilon\}$

Let $x_n \to x_0,\ f(x_n) = \frac{1}{x_n} \to \frac{1}{x_0} = f(x_0)$

$x_0 \epsilon (0, \infty) = \mathbb{R}^+, x_0 \neq 0, x_0 \epsilon \mathbb{R}^+$

Example 4: let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) \begin{cases} 2 \ \ if\ x\ is\ rational \\ 3 \ \ if\ x\ is\ irrational \end{cases}$$

Theorem 5.4:

Let $(x, d), (x', d')\ and\ (x'', d'')$ be metric spaces and let $f: x \to x'$ be continuous at x_0, cont. at x_0, $g: x' \to x''$ be cont. at $f(x_0)$ then $gof: x \to x''$ is cont. at x_0

$$x \xrightarrow{f} x' \xrightarrow{g} x''$$

Proof: let $< x_n >$ be a seq. in x s.t. $x_n \to x_0$

T.P. $(gof)(x_n) \to (gof)(x_0)$

$\because x_n \to x_0$ and f is cont. at x_0

$\therefore f(x_n) \to f(x_0), \quad < f(x_n) > a \quad seq.\ in\ x'$

$\because g\ is\ cont. at\ f(x_0)$

$\therefore g\big(f(x_n)\big) \to g\big(f(x_0)\big)$

$(gof)(x_n) \to (gof)(x_0)$

$\therefore gof \quad is \quad cont.$

Definition:

Let (x, d) be a metric space, the mapping $f: x \to \mathbb{R}$ is called real valued mapping.

Theorem 5.5:

Let $f, g: x \to \mathbb{R}$ be real valued mappings, if f and g cont. at x_0, then

1) $f \mp g$ is cont. at x_0.

2) $f.g$ is cont. at x_0.

3) $\frac{f}{g}$ is cont. at x_0.

4) cf is cont. at x_0, $\forall c \epsilon \mathbb{R}$.

5) $|f|$ is cont. at x_0.

Proof: 3) $\frac{f}{g} : x \to \mathbb{R}$

Let $< x_n >$ be a seq. in x s.t. $x_n \to x_0$

T.P. $\frac{f}{g}(x_n) \to \frac{f}{g}(x_0)$

$$\frac{f}{g}(x_n) = \frac{f(x_n)}{g(x_n)}$$

$\because f \; and \; g \; are \; cont.$ at $x_0 \Rightarrow f(x_n) \to f(x_0)$

And $g(x_n) \to g(x_0)$

$\left.\begin{matrix} a_n \to a_0 \\ b_n \to b_0 \end{matrix}\right\} \frac{a_n}{b_n} \to \frac{a_0}{b_0}$

Hence $\frac{f(x_n)}{g(x_n)} \to \frac{f(x_0)}{g(x_0)}$

Definition:

Let $f: x \to x'$ be a mapping we say that f is uniformly continuous, id $\forall x, y \epsilon X$.

$\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon)$ s.t. if $d(x, y) < \delta$ then $d'\big(f(x), f(y)\big) < \epsilon$

$\forall x, y \epsilon X$.

Theorem 5.8:

Every uniformly continuous function on an interval is continuous on that interval.

Remark: the converse of theorem 5.8 in general is not true as shown by the following example:

Ex.: let f defined on (0,1] as follows

$$f(x) = \frac{1}{x}, x\epsilon(0,1]$$

$\therefore f$ is cont.

For any $\delta > 0,\ \exists n\epsilon z^+$ s.t. $\frac{1}{n} < \delta$

Let $x = \frac{1}{n}\, and\ y = \frac{1}{2n}, \Rightarrow x, y\epsilon(0,1]$

$$|x - y| = \left|\frac{1}{n} - \frac{1}{2n}\right| = \left|\frac{1}{2n}\right| = \frac{1}{2n} < \frac{1}{n} < \delta$$

And

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = |n - 2n| = |-n| = n > 1$$

Hence $\delta > 0\ , \exists x, y\ \epsilon(0,1]$ s.t.

$|f(x) - f(y)| > \epsilon\ whenever\ |x - y| < \delta$

Hence $f(x) = \frac{1}{x}$ is not uniform continuous in (0,1]

$\therefore f(x) = \frac{1}{x}$ is cont. in (0,1] but not uniform cont. in (0,1]