Def: Let (X,d) be a metric space. An open cover of a set $E \subseteq X$ is a collection $\{G_\alpha\}_{\alpha\in\Lambda}$ of open sets of X such that $E \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha$.

Def: A subset E of a metric space is called compact if every open cover of E contains a finite subcover.

i.e $E \subseteq \bigcup_{\alpha\in\Lambda} \Rightarrow \exists\ \alpha_1, \alpha_2, \dots, \alpha_n$ s.t $E \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

proposition: Every finite set is compact.

proof: Let $E = \{a_1, a_2, \dots, a_n\}$ be finite set and suppose $\{G_\alpha\}_{\alpha\in\Lambda}$ is open cover of E

$\ni\ E \subset \bigcup_{\alpha\in\Lambda} G_\alpha$

since $a_1 \in E \Rightarrow a_1 \in \bigcup_{\alpha\in\Lambda} G_\alpha$

$\therefore\ a_1 \in G_\alpha$ for some $\alpha \in \Lambda$, is call G_{α_1}

$\therefore\ a_1 \in G_{\alpha_1}$ also $a_2 \in G_{\alpha_2}$ --- $a_n \in G_{\alpha_n}$

$\therefore\ \{a_1, a_2, \dots, a_n\} \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$

$\therefore\ E \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$

$\therefore$ A finite set E is compact

Theorem: (Hiene - Borel)

Every bounded closed subset of $\mathbb{R}^n$ is compact

Remark: A subset of compact m.s. is not necessary to be compact set, consider that see the following example.

Ex: Let $(\mathbb{R}, d)$ be ausual m.s.

By Hiene - Borel, $[0,1]$ is compact set

But $(0,1) \subset [0,1]$ is not compact set.

* The following theorem give the condition to make the subset of compact is compact.

Theorem: Every closed subset of compact is compact.

proof: Let (X, d) be compact m.s.

and S is closed.

then $A = X - S$ is open subset of X

T.P S is compact

Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be open cover of $S \ni S \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha$

since $X = S \cup A$

$$= \bigcup_{\alpha\in\Lambda} G_\alpha \cup A$$

$\therefore \bigcup G_\alpha \cup A$ is open cover of X

since X is compact

$$\Rightarrow X \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n} \cup A$$

since $A \cap S = \emptyset$

$\therefore S \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$

$\therefore S$ is compact

Theorem: Let (X,d), (Y,d') be m.s. $f: X \longrightarrow Y$ is cont. function. If S is compact subset of X then $f(S)$ is compact.

proof: Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $f(S) \ni f(S) \subseteq \bigcup_{\alpha\in\Lambda} \subseteq G_\alpha\}$

$$\Rightarrow S \subseteq f^{-1}\left(\bigcup_{\alpha\in\Lambda} \{G_\alpha\}\right)$$

$\therefore \; S \subseteq \bigcup f^{-1}\{G_\alpha\}$

since f is cont.

$\therefore \; f^{-1}\{G_\alpha\} = U_\alpha$ is open in X

$\therefore \; \bigcup_{\alpha \in \Lambda} U_\alpha$ is an open cover of S

$\therefore \; S \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$

since S is compact then $S \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \dots U_{\alpha_n}$

$$\Rightarrow f(S) \subseteq f(U_{\alpha_1}) \cup f(U_{\alpha_2}) \cup \dots \cup f(U_{\alpha_n})$$

$$\subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$$

$\therefore \; f(S)$ is compact

Theorem: If S is an infinite subset of compact m.s. X then S has limit point.

proof:

Suppose S has no L.p.

Then S is closed

Let $W = X - S$

then W is open

since S has no L.p. then $\exists \; U_x$

$\ni U_x \cap S = \{x\}$

$x \in U_x$, x is no L.p. of S

$S \subseteq \bigcup\limits_{x \in S} \{x\} \subseteq \bigcup\limits_{x \in S} U_x$

$X \subseteq W \cup \{U_x\}_{x \in S}$

since X is compact

$\Rightarrow X \subseteq W \cup U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$

and since $S = X \cap S$

$= S \cap (W \cup U_{x_1} \cup U_{x_2} \cdots \cup U_{x_n})$

$= (S \cap W) \cup (S \cap U_{x_1}) \cup (S \cap U_{x_2}) \cdots \cup (S \cap U_{x_n})$

$= \emptyset \cup \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\}$

$= \{x_1, x_2, \cdots, x_n\}$

$\therefore S$ is finite C!

$\therefore S$ has L.p.

Def: Let (X,d) be m.s. and E is subset of X, we say that E is bounded if $\exists$ real number M and a point $q \in X$ $\ni d(p,q) < M$, $\forall p \in E$

i.e ($\exists$ a ball $B_M \ni E \subseteq B_M$)

Theorem: Every compact subset of m.s. is bounded.

proof: Let (X,d) be a m.s. and Y be compact subset of X

$B_n(x_0) = \{ x_0 \in X : d(x, x_0) < n \}$

B_n is open subset of X

we claim $\{B_n\}$ is open cover of Y

Let $y_0 \in Y$, $\exists\ n \in N \ni d(y_0, x_0) < n$

$y_0 \in B_n(x_0) \Rightarrow y_0 \in \bigcup_{n=1}^{\infty} B_n$

$\therefore Y \subseteq \bigcup_{n=1}^{\infty} B_n$

$\therefore \{B_n\}$ is open cover of Y, and

since Y is compact

then $Y \subseteq B_{n_1} \cup B_{n_2} \cup \cdots \cup B_{n_k}$

Let $m = \max\{n_1, n_2, \ldots, n_k\}$

$\therefore Y \subseteq B_m$

$\therefore Y$ is compact.

Remark: The converse of above theorem is not necessary to be true. consider the following example

Ex: Let (R, d) be usual m.s. then $(0,1)$ is bounded in $(\mathbb{R}, d)$ but not compact.