2-5 Operators

The operator defines to be "a mathematical entity when acts on a wave function turn it to another function". i.e.

$$\hat{A}\Psi = \varphi \qquad \text{………… (2-11)}$$

Example 1: $\psi = x^3$, $\hat{A} = \mathrm{x}$

$$\hat{A}\psi = x \cdot x^3 = x^4 = \phi$$

Example 2: $\hat{A} = \frac{\partial}{\partial x}$, $\psi = x^3$

$$\hat{A}\psi = \frac{\partial}{\partial x} \cdot x^3 = 3x^2 = \phi$$

2-6 Operator equation

Let's consider the following operator; $\hat{A}(x, \frac{\partial}{\partial x}) = \frac{\partial}{\partial x} x$, so for any function $\psi(x)$ one get;

$$\begin{aligned}(\frac{\partial}{\partial x} x)\, \psi(x) &= \frac{\partial}{\partial x}(x\, \psi(x)) \\ &= \psi(x) + x \frac{\partial \psi(x)}{\partial x} \\ &= (1 + x \frac{\partial}{\partial x})\, \psi(x)\end{aligned}$$

Since the last equation is valid for any function for x, thus one can omit ψ from both sides and get;

$$\frac{\partial}{\partial x} x = (1 + x \frac{\partial}{\partial x}) \qquad \text{………… (2-12)}$$

Equation (2-12) called the operator equation.

2-7 Eigen value equation

For each operator $\hat{A}$ there being a set of numbers (a_n) and a set of functions $\psi_n(x)$ defined by the following formula;

$$\hat{A}\psi_n(x) = a_n \psi_n(x) \qquad \text{...........(2-13)}$$

Where a_n are called the **eigenvalues** and $\psi_n(x)$ are the corresponding **eigenfunctions**. Thus, the eigenfunctions of an operator are those special functions that remain unaltered under the operation of an operator apart from multiplication by the eigenvalue.

Example: By using the eigen value equation show that the function $\psi_n(x) = e^{i4x}$ is an eigen function of the operator $\hat{A} = \frac{\partial}{\partial x}$

Solution:

Let $\hat{A} = \frac{\partial}{\partial x}$ and $\psi_n(x) = e^{i4x}$. So,

$$\hat{A}\psi_n(x) = a_n \psi_n(x)$$

$$\hat{A}\psi_n(x) = \frac{\partial}{\partial x}(e^{i4x})$$

$$= i4e^{i4x}$$

$$= a_n \psi_n(x)$$

Hence, $a_n = i4$ is the eigen value and $\psi_n(x) = e^{i4x}$ is an eigen function.

Example: By using the eigen value equation show that the function $\hat{A} = -\frac{\partial^2}{\partial x^2}$ function of the operator eigen is an $\psi_n(x) = \cos(4x)$

Solution:

$$\hat{A}\psi_n(x) = a_n \psi_n(x)$$

$$\hat{A}\psi_n(x) = -\frac{\partial^2}{\partial x^2}\cos(4x)$$

$$4\frac{\partial}{\partial x}\sin(4x) = 16\cos(4x)$$

$$-\frac{\partial^2}{\partial x^2}(\cos(4x)) = 16\cos(4x)$$

$$\hat{A}\psi_n(x) = 16\cos(4x)$$

Hence, $a_n = 16$, $\psi_n(x) = \cos(4x)$ and so $\psi_n(x)$ remain unchanged, thus $\psi_n(x) = \cos(4x)$ is an eigen function for $\hat{A} = -\frac{\partial^2}{\partial x^2}$.

Example: By using the eigen value equation show that the function $\psi_n(x) = \sin(6x)$ is an eigen function of the operator . $\hat{A} = -\frac{\partial^2}{\partial x^2}$

Solution:

$$\hat{A}\psi_n(x) = a_n\psi_n(x)$$

$$\hat{A}\psi_n = -\frac{\partial^2}{\partial x^2}\sin(6x)$$

$$-6\frac{\partial}{\partial x}\cos(6x) = 36\sin(6x)$$

$$-\frac{\partial^2}{\partial x^2}\sin(6x) = 36\sin(6x)$$

$$\hat{A}\psi_n(x) = 36\sin(6x)$$

Hence, $a_n = 36$ and $\psi_n(x) = \sin(6x)$. The $\psi_n(x)$ remain unchanged, thus $\psi_n(x) = \sin(6x)$ is an eigen function for $\hat{A} = -\frac{\partial^2}{\partial x^2}$.

Example: prove that the function $\psi = Ae^{-\alpha x}$ is the eigen function of the operator $\hat{F} = \frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{2\alpha}{x}$ where the A, α are constants.

Solution:

$$\hat{F}\psi = \frac{d^2}{dx^2}(Ae^{-\alpha x}) + \frac{2}{x}\frac{d}{dx}(Ae^{-\alpha x}) + \frac{2\alpha}{x}(Ae^{-\alpha x})$$

$$\hat{F}\psi = \alpha^2 Ae^{-\alpha x} + \frac{2}{x}\frac{d}{dx}(-\alpha Ae^{-\alpha x}) + \frac{2\alpha}{x}(Ae^{-\alpha x})$$

$$\hat{F}\psi = \left(\alpha^2 - \frac{2\alpha}{x} + \frac{2\alpha}{x}\right) Ae^{-\alpha x}$$

$$\hat{F}\psi = \alpha^2 Ae^{-\alpha x}$$

$$\hat{F}\psi = \alpha^2\psi$$

It is seen that the function ψ is an eigen function with an eigen value α^2.

2-8 Operators properties

Operator has important properties namely the linearity, commutation and Hermitian.

1) **Linearity:** the operator $\hat{A}$ is said to be a linear operator if it satisfy the following conditions;

   i. $\hat{A}(\psi_1 + \psi_2) = \hat{A}\psi_1 + \hat{A}\psi_2$

   ii. $\hat{A}(a\psi) = a\hat{A}\psi$ where a is a constant

2) **Commutation:** the commutation relation between two operators $\hat{A}$ and $\hat{B}$ is define as;

$$\hat{C} = [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

Where $\hat{C}$ is called the commutator operator.

i. If $\hat{C} = 0 \Rightarrow [\hat{A}, \hat{B}] = 0 \Rightarrow \hat{A}\hat{B} = \hat{B}\hat{A}$

The operators in this case called **commute operators**.

ii. If $\hat{C} = 1$

The operators $\hat{C} =$ in this case called **unit operator**.

iii. If $\hat{C} \neq 0 \Rightarrow [\hat{A}, \hat{B}] \neq 0 \Rightarrow \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0$

The operators in this case called **not commute operators**.

Example: Prove that the operator $[\frac{\partial}{\partial x}, x]$ is a unit operator.

<u>Solution:</u>

$$\hat{C} = [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$\hat{C} = [\frac{\partial}{\partial x}, x] = \frac{\partial}{\partial x} x - x\frac{\partial}{\partial x}$$

Multiply the both sides by $\psi(x)$ one get;

$$\hat{C}\psi(x)=\left\{\frac{\partial}{\partial x}x-x\frac{\partial}{\partial x}\right\}\psi(x)$$

$$=\frac{\partial}{\partial x}x(\psi(x))-x\frac{\partial}{\partial x}(\psi(x))$$

$$=\frac{\partial}{\partial x}(x\psi(x))-x\frac{\partial\psi(x)}{\partial x}$$

$$=\psi(x)+x\frac{\partial\psi(x)}{\partial x}-x\frac{\partial\psi(x)}{\partial x}$$

$$\hat{C}\psi(x)=\psi(x)$$

$$\hat{C}=1$$

H.W Prove that; $\hat{C}=[x\ ,\ \frac{\partial}{\partial x}]=-1$.

Example: Show that; $[\hat{x},\hat{p}_x]=i\hbar$.

Solution:

$$\hat{c}=[\hat{x},\hat{p}_x]$$

$$\hat{c}=\hat{x}\,\hat{p}_x-\hat{p}_x\hat{x}$$

$$\hat{c}=\hat{x}(-i\hbar\frac{\partial}{\partial x})+i\hbar\frac{\partial}{\partial x}(\hat{x})$$

$$\hat{c}\psi(x)=\left\{\hat{x}(-i\hbar\frac{\partial}{\partial x})+i\hbar\frac{\partial}{\partial x}(\hat{x})\right\}\psi(x)$$

$$\hat{c}\psi(x)=\hat{x}(-i\hbar\frac{\partial\psi(x)}{\partial x})+i\hbar\frac{\partial}{\partial x}\hat{x}\ \psi(x)$$

$$\hat{c}\psi(x)=-i\hbar\,\hat{x}\frac{\partial\psi(x)}{\partial x})+i\hbar\psi(x)+i\hbar\,\hat{x}\frac{\partial\psi(x)}{\partial x}$$

$$\hat{c}\psi(x)=i\hbar\ \psi(x)$$

$$\hat{c}=i\hbar$$

H.W. Prove that $[\hat{p}_x,\hat{x}]=-i\hbar$