Solved Exercises

1) Show that $[p_x, V(x)] = -i\hbar \dfrac{\partial V(x)}{\partial x}$.

Solution:

$$[p_x, V(x)] = (p_x V(x) - V(x)\, p_x)$$

$$[p_x, V(x)]\psi(x) = (p_x V(x) - V(x)\, p_x)\psi(x)$$

$$= p_x V(x)\psi(x) - V(x)\, p_x \psi(x)$$

$$\because \quad p_x = -i\hbar \frac{\partial}{\partial x}$$

$$= -i\hbar \frac{\partial}{\partial x}(V(x)\,\psi(x)) - V(x)(-i\hbar \frac{\partial}{\partial x})\psi(x)$$

$$= -i\hbar V(x) \frac{\partial \psi(x)}{\partial x} - i\hbar \psi(x) \frac{\partial V(x)}{\partial x} + i\hbar\, V(x) \frac{\partial \psi(x)}{\partial x}$$

$$= -i\hbar \frac{\partial V(x)}{\partial x}\, \psi(x)$$

$$\therefore \quad [p_x, V(x)] = -i\hbar \frac{\partial V(x)}{\partial x}$$

2) Prove that the Hamiltonian operator of the free particle $\hat{H} = \dfrac{-\hbar^2}{2m}\dfrac{d^2}{dx^2}$ is the Hermitian operator.

Solution:

$$\int_{-\infty}^{\infty} \psi_n^* \hat{A} \psi_m \, dx = \int_{-\infty}^{\infty} \psi_m (\hat{A}\psi_n)^* dx$$

$$\int_{-\infty}^{\infty} \psi_n^* (\frac{-\hbar^2}{2m}\frac{d^2}{dx^2})\psi_m \, dx = \frac{-\hbar^2}{2m}\int_{-\infty}^{\infty} \psi_n^* \frac{d^2\psi_m}{dx^2}\, dx$$

Let; $u = \psi_n$ and $dv = \dfrac{d^2\psi_m}{dx^2}\, dx$, so; $du = \dfrac{d\psi_n}{dx}\, dx$ and $v = \dfrac{d}{dx}\psi_m$. Hence;

$$\int_{-\infty}^{\infty} u\,dv = uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du$$

$$= \frac{-\hbar^2}{2m}\psi_n^* \frac{d}{dx}\psi_n^* + \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} (\frac{d\psi_m}{dx})\,(\frac{d\psi_n^*}{dx})\,dx$$

$$= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} (\frac{d\psi_m}{dx})\,(\frac{d\psi_n^*}{dx})\,dx$$

Now let; $u = \frac{d\psi_n^*}{dx}$ and $dv = \frac{d\psi_m}{dx}\,dx$, so; $du = \frac{d^2\psi_n^*}{dx^2}\,dx$ and $v = \psi_m$. Thus;

$$\frac{-\hbar^2}{2m}\psi_m \frac{d\psi_n^*}{dx}\Bigg|_{-\infty}^{\infty} - \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\psi_m \frac{d^2\psi_n^*}{dx^2}\,dx$$

$$= \int_{-\infty}^{\infty}\psi_m\,(-\frac{\hbar^2}{2m}\frac{d^2\psi_n^*}{dx^2})\,dx$$

$$= \int_{-\infty}^{\infty}\psi_m\,(-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2})^*\,dx$$

$$= \int_{-\infty}^{\infty}\psi_m\,(\hat{H}\psi_n)^*\,dx$$

3) Show that $[\hat{H}\,,\,\hat{x}] = \frac{-i\hbar}{m}\hat{p}_x$

Solution:

$$[\hat{H}\,,\,\hat{x}] = \hat{H}\,\hat{x} - \hat{x}\,\hat{H}$$

Where $\hat{H} = \frac{-\hbar^2}{2m}\nabla^2 + V(x,y,z)$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

$$[\hat{H}\,,\,\hat{x}]\psi = (\hat{H}\,\hat{x} - \hat{x}\,\hat{H})\,\psi$$

$$\left[\hat{H}\,,\,\hat{x}\right]\psi = \left\{(\frac{-\hbar^2}{2m}\nabla^2 + V(x,y,z))x - x(\frac{-\hbar^2}{2m}\nabla^2 + V(x,y,z))\right\}\psi$$

$$=-\frac{\hbar^2}{2m}\nabla^2 x\psi + V(x,y,z)\, x\psi + x\frac{\hbar^2}{2m}\nabla^2\psi - xV(x,y,z)\psi$$

$$=-\frac{\hbar^2}{2m}\nabla^2 x\psi + x\frac{\hbar^2}{2m}\nabla^2\psi$$

$$\nabla^2 x\psi = \nabla(\nabla x\,\psi)$$

$$=\nabla(x\nabla\psi + \psi)$$

$$= x\nabla^2\psi + \nabla\psi + \nabla\psi$$

$$= x\nabla^2\psi + 2\nabla\psi$$

$$=-x\frac{\hbar^2}{2m}\nabla^2\psi - \frac{\hbar^2}{2m}2\nabla\psi + x\frac{\hbar^2}{2m}\nabla^2\psi$$

$$\left[\hat{H}\,,\,\hat{x}\right]\psi = -\frac{\hbar^2}{m}\nabla\psi$$

$$\left[\hat{H}\,,\,\hat{x}\right] = -\frac{\hbar^2}{m}\nabla \quad \Rightarrow \quad -\frac{i\hbar}{m}\frac{\hbar}{i}\nabla = -\frac{i\hbar}{m}\hat{p}_x$$

4) Find p_d , A, $\langle x\rangle$, $(\Delta x)^2$, $\langle p\rangle$,$(\Delta p)^2$, $\Delta p\,\Delta x$ For the following one-dimensional wave function;

$$\psi(x) = A\, e^{-\frac{(x-x_\circ)^2}{2a^2}}\, e^{i\,p_\circ x/\hbar}$$

Solution:

$$\psi(x) = A\, e^{-\frac{(x-x_\circ)^2}{2a^2}}\, e^{i\,p_\circ x/\hbar} \quad , \quad \psi^*(x) = A\, e^{-\frac{(x-x_\circ)^2}{2a^2}}\, e^{-i\,p_\circ x/\hbar}$$

$$p_d = |\psi(x)|^2 = AA^*\, e^{-\frac{(x-x_\circ)^2}{a^2}}$$

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1$$

$$A\,A^* \int_{-\infty}^{\infty} e^{-\frac{(x-x_\circ)}{a^2}}\, dx = 1$$

Let $(x - x_\circ) = z \Rightarrow dx = dz$, so;

$$AA^* \int_{-\infty}^{\infty} e^{-\frac{z^2}{a^2}} dz = 1$$

$$AA^* a\sqrt{\pi} = 1$$

$$AA^* = \frac{1}{a\sqrt{\pi}}$$

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^* x\, \psi\, dx$$

$$\langle x\rangle = \int_{-\infty}^{\infty} A\, e^{-\frac{(x-x_\circ)^2}{2a^2}} e^{-i p_\circ x/\hbar} \;x\; A^*\, e^{-\frac{(x-x_\circ)^2}{2a^2}} e^{i p_\circ x/\hbar}\, dx$$

$$\langle x\rangle = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} x\; e^{-\frac{(x-x_\circ)^2}{a^2}}\, dx$$

Let $(x - x_\circ) = z \Rightarrow x = z + x_\circ \quad dx = dz$, so;

$$\langle x\rangle = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} (z + x_\circ)\; e^{-\frac{z^2}{a^2}}\, dz$$

$$\langle x\rangle = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{a^2}} dz + \frac{x_\circ}{a\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{a^2}}\, dz$$

$$\langle x\rangle = \frac{x_\circ}{a\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{a^2}}\, dz = \frac{x_\circ}{a\sqrt{\pi}}\, a\sqrt{\pi}$$

$$\therefore \quad \langle x\rangle = x_\circ$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} \psi^* x^2\, \psi\, dx$$

$$\langle x\rangle = \int_{-\infty}^{\infty} A\, e^{-\frac{(x-x_\circ)^2}{2a^2}} e^{-i p_\circ x/\hbar} \;x^2\; A^*\, e^{-\frac{(x-x_\circ)^2}{2a^2}} e^{i p_\circ x/\hbar}\, dx$$

$$\langle x\rangle = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 \ e^{-\frac{(x-x_\circ)^2}{a^2}} \ dx$$

Let $x - x_\circ = z \qquad \Rightarrow \qquad x^2 = z^2 + 2x_\circ z + x_\circ^2 \qquad dx = dz$

$$\langle x\rangle = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} (z^2 + 2x_\circ z + x_\circ^2) \ e^{-\frac{(x-x_\circ)^2}{a^2}} \ dz$$

$$\langle x^2\rangle = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} z^2 \ e^{-\frac{z^2}{a^2}} dz + \frac{2x_\circ}{a\sqrt{\pi}} \int_{-\infty}^{\infty} z \, e^{-\frac{z^2}{a^2}} dz + \frac{2x_\circ^2}{a\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{a^2}} dz$$

$$\langle x^2\rangle = \frac{1}{a\sqrt{\pi}} \cdot \frac{1}{2}\sqrt{a^6}\sqrt{\pi} + \frac{x_\circ^2}{a\sqrt{\pi}} \cdot a\sqrt{\pi}$$

$$\langle x^2\rangle = \frac{1}{2}a^2 + x_\circ^2$$

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$$

$$(\Delta x)^2 = \frac{1}{2}a^2 + x_\circ^2 - x_\circ^2$$

$$\therefore \ (\Delta x)^2 = \frac{1}{2}a^2$$

$$\langle p_x\rangle = \int \psi^* p_x \psi \ dx$$

Since; $\psi = A\, e^{-\frac{(x-x_\circ)^2}{2a^2} + \frac{i p_\circ x}{\hbar}}$ so; $\psi = A\, e^{-\frac{(x-x_\circ)^2}{2a^2} - \frac{i p_\circ x}{\hbar}}$ and remember; $p_x = -i\hbar \frac{\partial}{\partial x}$,hence

$$\langle p_x\rangle = -i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial}{\partial x}\psi \ dx$$

$$\frac{\partial \psi}{\partial x} = A\, e^{-\frac{(x-x_\circ)^2}{2a^2} + \frac{i p_\circ x}{\hbar}} \left[-\frac{(x-x_\circ)}{a^2} + \frac{i p_\circ}{\hbar}\right]$$

$$\langle p_x\rangle = -i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial}{\partial x}\psi \ dx$$

$$= i\hbar \int_{-\infty}^{\infty} A^* \, e^{-\frac{(x-x_\circ)^2}{2a^2} - \frac{i p_\circ x}{\hbar}} \quad A\, e^{-\frac{(x-x_\circ)^2}{2a^2} + \frac{i p_\circ x}{\hbar}} [-\frac{(x-x_\circ)}{a^2} + \frac{i p_\circ}{\hbar}]\, dx$$

$$= -A^* A i\hbar \int_{-\infty}^{\infty} e^{-\frac{(x-x_\circ)^2}{a^2}} [-\frac{(x-x_\circ)}{a^2} + \frac{i p_\circ}{\hbar}]\, dx$$

Let $x - x_\circ = z \Rightarrow x = z + x_\circ \Rightarrow dx = dz$

$$= -A^* A i\hbar \int_{-\infty}^{\infty} e^{-\frac{z^2}{a^2}} [-\frac{z}{a^2} + \frac{i p_\circ}{\hbar}]\, dz$$

$$= \frac{A^* A i\hbar}{a^2} \int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{a^2}} + A^* A\, p_\circ \int_{-\infty}^{\infty} e^{-\frac{z^2}{a^2}}\, dz$$

$$= 0 + \frac{1}{a\sqrt{\pi}}\, p_\circ \int_{-\infty}^{\infty} e^{-\frac{z^2}{a^2}}\, dz$$

$$= \frac{1}{a\sqrt{\pi}}\, p_\circ\, a\sqrt{\pi}$$

$$= p_\circ$$

$$\langle p_x \rangle^2 = p_\circ^2$$

$$\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi^* \frac{\partial^2}{\partial x^2} \psi \; dx$$

$$\frac{\partial \psi}{\partial x} = A\, e^{-\frac{(x-x_\circ)^2}{2a^2} + \frac{i p_\circ x}{\hbar}} [-\frac{(x-x_\circ)}{a^2} + \frac{i p_\circ}{\hbar}]$$

$$\frac{\partial^2 \psi}{\partial x^2} = A\, e^{-\frac{(x-x_\circ)^2}{2a^2} + \frac{i p_\circ x}{\hbar}} [-\frac{(x-x_\circ)}{a^2} + \frac{i p_\circ}{\hbar}]^2 - \frac{A}{a^2}\, e^{-\frac{(x-x_\circ)^2}{2a^2} + \frac{i p_\circ x}{\hbar}}$$

$$< p^2 > = -\hbar^2 \int_{-\infty}^{\infty} A^*\, e^{-\frac{(x-x_\circ)^2}{2a^2} - \frac{i p_\circ x}{\hbar}} \left(A\, e^{-\frac{(x-x_\circ)^2}{2a^2} + \frac{i p_\circ x}{\hbar}} [-\frac{(x-x_\circ)}{a^2} + \frac{i p_\circ}{\hbar}]^2 - \frac{A}{a^2}\, e^{-\frac{(x-x_\circ)^2}{2a^2} + \frac{i p_\circ x}{\hbar}} \right) dx$$

$$<p^2>=-\hbar^2\int_{-\infty}^{\infty}A^*A\,e^{-\frac{(x-x_\circ)^2}{a^2}}[-\frac{(x-x_\circ)}{a^2}+\frac{i\,p_\circ}{\hbar}]^2+\frac{A^*A\,\hbar^2}{a^2}\int_{-\infty}^{\infty}e^{-\frac{(x-x_\circ)^2}{a^2}}\,dx$$

Let $(x-x_\circ)=z \Rightarrow x=z+x_\circ \Rightarrow dx=dz$

$$<p^2>=-\hbar^2\int_{-\infty}^{\infty}A^*A\,e^{-\frac{z^2}{a^2}}[-\frac{z}{a^2}+\frac{i\,p_\circ}{\hbar}]^2+\frac{A^*A\,\hbar^2}{a^2}\int_{-\infty}^{\infty}e^{-\frac{z^2}{a^2}}\,dx$$

$$<p^2>=-\hbar^2\int_{-\infty}^{\infty}A^*A\,e^{-\frac{z^2}{a^2}}[-\frac{z}{a^2}+\frac{i\,p_\circ}{\hbar}]^2+\frac{A^*A\,\hbar^2}{a^2}\int_{-\infty}^{\infty}e^{-\frac{z^2}{a^2}}\,dx$$

$$<p^2>=\frac{-A^*A\,\hbar^2}{a^4}\int_{-\infty}^{\infty}z^2\,e^{-\frac{z^2}{a^2}}\,dz+\frac{2A^*A\,i\hbar\,p_\circ}{a^2}\int_{-\infty}^{\infty}z\,e^{-\frac{z^2}{a^2}}\,dz+A^*A\,p_\circ^2\int_{-\infty}^{\infty}e^{-\frac{z^2}{a^2}}+\frac{A^*A\,\hbar^2}{a^2}\int_{-\infty}^{\infty}e^{-\frac{z^2}{a^2}}\,dz$$

$$<p^2>=\frac{-\hbar^2}{a^5\sqrt{\pi}}\int_{-\infty}^{\infty}z^2\,e^{-\frac{z^2}{a^2}}\,dz+\frac{2\,i\hbar\,p_\circ}{a^3\sqrt{\pi}}\int_{-\infty}^{\infty}z\,e^{-\frac{z^2}{a^2}}\,dz+\frac{p_\circ^2}{a\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\frac{z^2}{a^2}}\,dz+\frac{\hbar^2}{a^3\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\frac{z^2}{a^2}}\,dz$$

$$=\frac{-\hbar^2}{2a^5\sqrt{\pi}}\cdot\sqrt{a^6}\sqrt{\pi}+0+\frac{p_\circ^2}{a\sqrt{\pi}}\cdot a\sqrt{\pi}+\frac{\hbar^2}{a^3\sqrt{\pi}}\cdot\sqrt{\pi}\sqrt{a^2}$$

$$=-\frac{\hbar^2}{2a^2}+p_\circ^2+\frac{\hbar^2}{a^2}$$

$$=p_\circ^2+\frac{\hbar^2}{2a^2}$$

$$(\Delta p)^2=\langle p^2\rangle-\langle p\rangle^2$$

$$=p_\circ^2+\frac{\hbar^2}{2a^2}-p_\circ^2$$

$$(\Delta p)^2=\frac{\hbar^2}{2a^2}$$

$$\Delta p\Delta x=\frac{\hbar}{2}$$