Solved Problem

1) A particle descript by the following wave function; $\psi(x)=\sqrt{\frac{2}{a}}\sin(n\pi x/a)$. Find the following; i) $\langle p_x\rangle$, ii) $\langle p_x^2\rangle$, iii) $\langle x\rangle$, iv) $\langle x^2\rangle$ and v) $\Delta p\,\Delta x$.

Solution:

i) $$\langle p_x\rangle=\int_{-\infty}^{\infty}\psi_n(x)\,\hat{p}_x\,\psi_n(x)\,dx$$

$$=\int_0^a\sqrt{\tfrac{2}{a}}\sin(n\pi x/a)(-i\hbar\frac{\partial}{\partial x})\sqrt{\tfrac{2}{a}}\sin(n\pi x/a)\,dx$$

$$=\frac{-2i\hbar}{a}\int_0^a\sin(n\pi x/a)\cdot\frac{n\pi}{a}\cdot\cos(n\pi x/a)\,dx$$

$$=\frac{-2i\hbar n\pi}{a^2}\int_0^a\sin(n\pi x/a)\cos(n\pi x/a)\,dx$$

$$=\frac{-i\hbar}{a}\sin^2(n\pi x/a)\Big|_0^a$$

$$=0$$

The physical meaning of this result is that the mumentum of the particles that moves in –x-axis is exactle similar to that of the particles which moves in the +x-axis.

ii)

$$\langle p_x^2\rangle=\int_0^a\sqrt{\tfrac{2}{a}}\sin(n\pi x/a)(-\hbar^2\frac{\partial^2}{\partial x^2})\sqrt{\tfrac{2}{a}}\sin(n\pi x/a)\,dx$$

$$=\frac{-2\hbar^2}{a}\int_0^a\sin(n\pi x/a)\frac{\partial^2}{\partial x^2}\sin(n\pi x/a)\,dx$$

$$=\frac{-2\hbar^2}{a}\int_0^a\sin(n\pi x/a)\frac{\partial}{\partial x}\frac{n\pi}{a}\cos(n\pi x/a)\,dx$$

$$=\frac{-2n\pi\hbar^2}{a^2}\int_0^a\sin(n\pi x/a)\frac{\partial}{\partial x}\cos(n\pi x/a)\,dx$$

$$=\frac{2n^2\pi^2\hbar^2}{a^3}\int_0^a\sin(n\pi x/a)\sin(n\pi x/a)\,dx\quad=\frac{2n^2\pi^2\hbar^2}{a^3}\int_0^a\sin^2(n\pi x/a)\,dx$$

$$\langle p^2\rangle = \frac{2n^2\pi^2\hbar^2}{a^3}\left(\frac{x}{2} - \frac{\sin 2\frac{n\pi x}{a}}{4\frac{n\pi}{a}}\right)\Bigg|_0^a$$

$$\langle p^2\rangle = \frac{2n^2\pi^2\hbar^2}{a^3}\left(\ \frac{a}{2}\ \right) = \frac{n^2\pi^2\hbar^2}{a^2}$$

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{n^2\pi^2\hbar^2}{a^2} - 0$$

$$= \frac{n^2\pi^2\hbar^2}{a^2}$$

$$\therefore \quad (\Delta p) = \frac{n\pi\hbar}{a}$$

iii)

$$\langle x\rangle = \int_0^a \psi_n(x)\ x\psi_n(x)\ dx$$

$$= \int_0^a \sqrt{\tfrac{2}{a}}\sin(n\pi x/a)\ \ x\ \ \sqrt{\tfrac{2}{a}}\sin(n\pi x/a)\ dx$$

$$= \frac{2}{a}\int_0^a x\ sin^2(n\pi x/a)\ dx$$

This integration is very simple and valuable

$$= \frac{2}{a}\left(\frac{x^2}{4} - \frac{\sin 2\frac{n\pi x}{a}}{4\frac{n\pi}{a}} - \frac{\cos 2\frac{n\pi x}{a}}{8(\frac{n\pi}{a})^2}\right)\Bigg|_0^a$$

$$\langle x\rangle = \frac{a}{2} \quad \Rightarrow \quad \langle x\rangle^2 = \frac{a^2}{4}$$

It indicates the presence of the particle in half of the left or right box with the same probability

Now we find $\langle x^2\rangle$

$$\langle x^2\rangle = \int_0^a \psi_n(x)\ x^2\,\psi_n(x)\ dx$$

$$=\int_0^a \sqrt{\tfrac{2}{a}}\sin(n\pi x/a)\;\; x^2 \sqrt{\tfrac{2}{a}}\sin(n\pi x/a)\; dx$$

$$=\frac{2}{a}\int_0^a x^2 \sin^2(n\pi x/a)\;\; dx$$

This integration is very simple and valuable

$$=\left(\frac{2}{a}\right)\times\left[\frac{x^2}{6}-\left(\frac{x^2}{4(\frac{n\pi}{a})}-\frac{1}{8\left(\frac{n\pi}{a}\right)^3}\right)\sin(\frac{2\pi x}{a})-\frac{x cos(2\pi x/a)}{4\left(\frac{n\pi}{a}\right)^2}\right]\Bigg|_0^a$$

$$\langle x^2\rangle=\frac{a^2}{3}-\frac{a^2}{2n^2\pi^2}$$

$$\because\quad (\Delta x)^2=\langle x^2\rangle-\langle x\rangle^2$$

$$=\frac{a^2}{3}-\frac{a^2}{2n^2\pi^2}-\frac{a^2}{4}$$

$$=a^2\left[\frac{1}{12}-\frac{1}{2n^2\pi^2}\right]$$

$$\therefore\quad \Delta x=a\left[\frac{1}{12}-\frac{1}{2n^2\pi^2}\right]$$

$$\Delta p\,\Delta x=\frac{n\pi\hbar}{a}-a\left[\frac{1}{12}-\frac{1}{2n^2\pi^2}\right]$$

$$\Delta p\,\Delta x=\hbar\left[\frac{n^2\pi^2}{12}-\frac{1}{2}\right]$$

The lowest value of the multiplication factor belongs to the ground state $n=1$

$$\Delta p\,\Delta x=0.567\,\hbar$$

This result is consistent with $\Delta p\,\Delta x\geq\hbar$

Q3) What is the energy for the particle described by the wave function $\psi(x)=\sqrt{\frac{2}{a}}\sin(n\pi\, x/a)$ and move along the interval $0\le x\le a$.

Solution:

$$\hat{A}\psi_n = a_n\psi_n$$
$$\hat{H}\psi_n = E_n\psi_n$$
$$=\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\cdot\sqrt{\tfrac{2}{a}}\sin(n\pi\, x/a)$$
$$=\frac{-\hbar^2}{2m}\frac{\partial}{\partial x}\{\sqrt{\tfrac{2}{a}}\cdot\frac{n\pi}{a}\cdot\cos(n\pi\, x/a)\}$$
$$=\frac{n^2\pi^2\hbar^2}{2ma^2}\cdot\sqrt{\tfrac{2}{a}}\sin(n\pi\, x/a)$$
$$=\frac{n^2\pi^2\hbar^2}{2ma^2}\psi_n$$
$$\therefore\quad E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}$$

Q4) What is the momentum square for the particle described by the wave function $\psi(x)=\sqrt{\frac{2}{a}}\sin(n\pi x/a)$ and move along the interval $0\le x\le a$.

Solution:

$$\hat{A}\psi_n = a_n\,\psi_n$$
$$\hat{p}^2\psi_n = p_n^2\,\psi_n$$
$$=-\hbar^2\,\frac{\partial^2}{\partial x^2}\,\sqrt{\tfrac{2}{a}}\,\sin(n\pi x/a)$$
$$=-\hbar^2\,\frac{\partial}{\partial x}\{\sqrt{\tfrac{2}{a}}\,\frac{n\pi}{a}\,\cos(n\pi x/a)\}$$
$$=\frac{n^2\pi^2\hbar^2}{a^2}\,\sqrt{\tfrac{2}{a}}\,\sin(n\pi x/a)$$
$$=\frac{n^2\pi^2\hbar^2}{a^2}\,\psi_n$$
$$\therefore\quad p_n^2=\frac{n^2\pi^2\hbar^2}{a^2}$$

Q5) Show that the wave function that described particle move in a potential box are orthogonal

Solution:

$$\int_0^a \psi_n^* \psi_m \, dx = 0 \qquad \text{Orthogonal} \quad (\text{متعامدة})$$

$$\int_0^a \psi_n^* \psi_m \, dx = \int_0^a \sqrt{\tfrac{2}{a}} \sin(n\pi\, x/a) \cdot \sqrt{\tfrac{2}{a}} \sin(m\pi\, x/a)\, dx$$

$$= \tfrac{2}{a} \int_0^a \sin(n\pi\, x/a) \cdot \sin(m\pi\, x/a)\, dx$$

وباستخدام العلاقة

$$\sin\alpha\, x \sin\beta\, x = \tfrac{1}{2}\{\cos(\alpha-\beta)\,x - \cos(\alpha+\beta)\,x\}$$

$$\tfrac{1}{a}\int_0^a \{\cos(n-m)\pi x/a - \cos(n+m)\pi x/a\}\, dx$$

$$\frac{1}{a}\{\frac{a}{(n-m)\pi}\sin(n-m)\pi\, x/a - \frac{a}{(n+m)\pi}\sin(n+m)\pi\, x/a\}\Big|_0^a$$

$$\frac{1}{a}\{\frac{a}{(n-m)\pi}\sin(n-m)\pi - \frac{a}{(n+m)\pi}\sin(n+m)\pi\}$$

$\because$ $(n-m)$ and $(n+m)$ are integer

$$\therefore \quad \int_0^a \psi_n^* \psi_m dx = 0$$

Q6) Show that the wave function that described particle move in a potential box are normalized

Solution:

$$\int_0^a \psi_n^* \psi_n \, dx = 1 \qquad \text{normalized} \quad (\text{عيارية})$$

$$\int_0^a \psi_n^* \psi_n \, dx = \int_0^a \sqrt{\tfrac{2}{a}} \sin(n\pi x/a) \cdot \sqrt{\tfrac{2}{a}} \sin(n\pi x/a) \; dx$$

$$= \frac{2}{a} \int_0^a \sin^2(n\pi x/a) \, dx$$

By using $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\therefore \quad = \frac{2}{a} \int_0^a \{\frac{1}{2} - \frac{1}{2}\cos(2n\pi x/a)\} \; dx$$

$$= \frac{1}{a} \int_0^a dx - \frac{1}{a} \int_0^a \cos(2n\pi x/a) \, dx$$

$$\frac{1}{a}(x)\Big|_0^a - \frac{1}{a}\frac{a}{2n\pi}\sin(\frac{2n\pi x}{a})\Big|_0^a$$

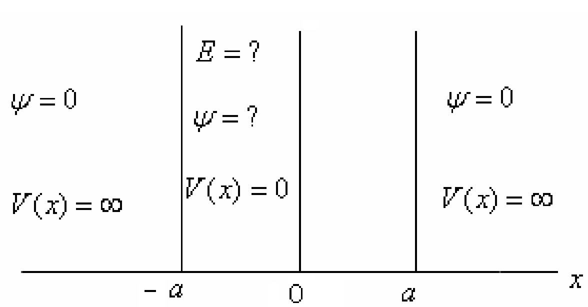
$$\frac{1}{a}(a-0) - \frac{1}{2n\pi}\sin(2n\pi)$$

$$\because \; n \; \text{is integer} \quad \Rightarrow \quad \sin 2n\pi = 0$$

$$\therefore \; \int_0^a \psi_n^* \psi_n dx = 1$$

Example: Consider a particle movement in a field of symmetrical potential as the shape

and described as follows:



$$V(x) = \begin{cases} 0, & |x| < a \\ \infty, & |x| > a \end{cases}$$

proved that:

$$\psi_n(x) = \begin{cases} A\sin(n\pi x/2a) & n \text{ is even} \\ B\cos(n\pi x/2a) & n \text{ is odd} \end{cases}$$

$$E_n = \frac{\pi^2\hbar^2 n^2}{8ma^2},\ A = B = \sqrt{\tfrac{1}{a}}$$

Solution :

$$\frac{d^2\psi(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi(x) = 0 \quad ، \quad V(x) = 0$$

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0 \quad ، \quad k^2 = \frac{2mE}{\hbar^2} \quad ، \quad E = \frac{\hbar^2 k^2}{2m}$$

$$\psi(x) = A\sin kx + B\cos kx$$

To find the two constant *A* and *B*, we use the two boundary conditions.

The wave function vanishes and becomes zero when $x = -a$ and when $x = a$

In the region *x = a*

$$A\sin ka + B\cos ka = 0$$

In the region *x = - a*

$$-A\sin ka + B\cos ka = 0$$

These two equations give us:

$$-A\sin ka = B\cos ka = 0$$

Where *A*, *B* can not be equal to zero because that means that ψ everywhere and so sin*ka*, cos*ka* they can not be zero at the same time for this reason. The only two possible solutions to the equation are:

a. either $A = 0$ and $\cos ka = 0$
b. or $B = 0$ and $\sin ka = 0$

These two results have the following meaning

$$ka = \frac{n\pi}{2}$$

Where *n* an odd number for case a and an even number of case b and so are the two possible solutions as they are.

When we compensate for the value *k* we get the wave function as well as the energy values of the particle.

$$\psi_n(x) = B\sin(n\pi x/2a) \qquad n \qquad \text{even number}$$

$$\psi_n(x) = A\cos(n\pi x/2a) \qquad n \qquad \text{odd number}$$

$$E = \frac{\pi^2\hbar^2 n^2}{8ma^2}$$

To find the constant *B* of the equation $\psi_n(x) = B\sin(n\pi x/2a)$ we use the normalize condition for any wave functions ie:

$$\int \psi_n^*(x)\psi_n(x)\; dx = 1$$

$$\int B^* \sin(n\pi x/2a)\; B\sin(n\pi x/2a)\; dx = 1$$

$$B^2 \int_{-a}^{a} \sin^2(n\pi x/2a)\; dx = 1$$

By used $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\frac{1}{2}B^2\left\{\int_{-a}^{a} dx - \int_{-a}^{a}\cos(n\pi x/a)\; dx\right\} = 1$$

$$\tfrac{1}{2}B^2\left\{x\Big|_{-a}^{a} - \left(\frac{a}{n\pi}\right)\sin(n\pi x/a)\Big|_{-a}^{a}\right\} = 1$$

$$\tfrac{1}{2}B^2\{(2a-0) - (a/n\pi)(0-0)\} = 1$$

$$\tfrac{1}{2}B^2\, 2a = 1 \longrightarrow \qquad C = \sqrt{\tfrac{1}{a}}$$

$$\therefore \quad \psi_n(x) = \sqrt{\tfrac{1}{a}}\sin(n\pi x/a)$$

In the same way you can find a constant *A*