Definition :-

Let (X,d) be a metric space a sequance $\langle a_n \rangle$ is called cauchy sequnce if for every $\epsilon > 0$, there is positive integer N, such that

$d(a_n, a_m) < \epsilon \quad \forall\, n \geq N$ and $m \geq N$.

Example :- let $(\mathbb{R}, d)$ be the usual metric space, show that a sequnce $\langle \frac{1}{n} \rangle$ is cauchy.

Solution :- 1. Let $\epsilon > 0$ be given

2. let N be the smallest positive integer $\ni N > \frac{2}{\epsilon}$

$\forall\, n \geq N, m \geq N.$

$\because n \geq N > \frac{2}{\epsilon} \Rightarrow \frac{1}{n} \leq \frac{1}{N} < \frac{\epsilon}{2} \Rightarrow \frac{1}{n} < \frac{\epsilon}{2} \Rightarrow |\frac{1}{n}| < \frac{\epsilon}{2}$

by the same way

$m \geq N > \frac{2}{\epsilon} \Rightarrow \frac{1}{m} \leq \frac{1}{N} < \frac{\epsilon}{2} \Rightarrow \frac{1}{m} < \frac{\epsilon}{2}$

$\Rightarrow |\frac{1}{m}| < \frac{\epsilon}{2}$

$\therefore d(a_n, a_m) = |\frac{1}{n} - \frac{1}{m}| \leq |\frac{1}{n}| + |\frac{1}{m}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\therefore d(a_n, a_m) < \epsilon$

$\therefore \langle \frac{1}{n} \rangle$ is cauchy sequnce.