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2. Iterative methods :-

a- Jacobi's method :-

Given n equations with n variables rearrange each equation to give a different variable and each equation to give a different variable and put in the form:

$X = TX + C$

Let, we have a system of linear equations:

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = b_1$$
$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 = b_2$$
$$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = b_3$$

Such that $a_{11} \neq a_{12} \neq a_{13}$

$$\Rightarrow \quad X_1 = [b_1 - (a_{12}X_2 + a_{13}X_3)] / a_{11}$$
$$X_2 = [b_2 - (a_{21}X_1 + a_{23}X_3)] / a_{22}$$
$$X_3 = [b_3 - (a_{31}X_1 + a_{32}X_2)] / a_{33}$$

A general form of the iterative formula defined by :-

$$X_1^{(K+1)} = [b_1 - (a_{12}X_2^{(K)} + a_{13}X_3^{(K)}] / a_{11}$$
$$X_2^{(K+1)} = [b_2 - (a_{21}X_1^{(K)} + a_{23}X_3^{(K)})] / a_{22}$$
$$X_3^{(K+1)} = [b_3 - (a_{31}X_1^{(K)} + a_{32}X_2^{(K)})] / a_{33}$$

$K = 0, 1, 2, \ldots$

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Example: By Jacobi's method find the solution of the following system:

$$x_1 + 8x_2 + 3x_3 = -4$$
$$-2x_1 - x_2 + 10x_3 = 9$$
$$10x_1 + 2x_2 - x_3 = 7$$

Use two loops and the intial value:

$x_1^{(0)} = 0,\ x_2^{(0)} = 0,\ x_3^{(0)} = 0$?

Solution:-

Firstly rearrange the equations in the system, we get:

$$\left.\begin{aligned} 10x_1 + 2x_2 - x_3 &= 7 \\ x_1 + 8x_2 + 3x_3 &= -4 \\ -2x_1 - x_2 + 10x_3 &= 9 \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} x_1^{(k+1)} &= \frac{-2}{10} x_2^{(k)} + \frac{1}{10} x_3^{(k)} + \frac{7}{10} \\ x_2^{(k+1)} &= -\frac{1}{8} x_1^{(k)} - \frac{3}{8} x_3^{(k)} - \frac{4}{8} \\ x_3^{(k+1)} &= \frac{2}{10} x_1^{(k)} + \frac{1}{10} x_2^{(k)} + \frac{9}{10} \end{aligned}\right\}$$

$k = 0, 1, 2, \ldots$

$k = 0$

$$x_1^{(1)} = \frac{-2}{10} x_2^{(0)} + \frac{1}{10} x_3^{(0)} + \frac{7}{10}$$
$$x_2^{(1)} = -\frac{1}{8} x_1^{(0)} - \frac{3}{8} x_3^{(0)} - \frac{4}{8} \quad , x_1^{(0)} = 0,\ x_2^{(0)} = 0, x_3^{(0)} = 0$$
$$x_3^{(1)} = \frac{2}{10} x_1^{(0)} + \frac{1}{10} x_2^{(0)} + \frac{9}{10}$$

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$$\Rightarrow x_1^{(1)} = 0.7 \,,\quad x_2^{(1)} = -0.5 \,,\quad x_3^{(1)} = 0.9$$

$K = 1$

$$x_1^{(2)} = \frac{-2}{10} x_2^{(1)} + \frac{1}{10} x_3^{(1)} + \frac{7}{10}$$

$$x_2^{(2)} = -\frac{1}{8} x_1^{(1)} - \frac{3}{8} x_3^{(1)} - \frac{4}{8}$$

$$x_3^{(2)} = \frac{2}{10} x_1^{(1)} + \frac{1}{10} x_2^{(1)} + \frac{9}{10}$$

$$\Rightarrow x_1^{(2)} = 0.89 \,,\quad x_2^{(2)} = -0.925 \,,\quad x_3^{(2)} = 0.99$$

Exercise:

Solve the following system of linear equations by using Jacobi's method:

$$4x_1 + x_2 - 2x_3 = 1$$

$$x_1 - 7x_2 + 10x_3 = 2$$

$$x_1 + 3x_2 - x_3 = 8$$

where $x_1^{(0)} = 1$, $x_2^{(0)} = 3$, $x_3^{(0)} = 2$, use two loops?