STUDY THE SOLUTION GROWTH OF SECOND AND HIGHER ORDER COMPLEX LINEAR DIFFERENTIAL EQUATIONS

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Abstract: *in this paper we shall prove that any nontrivial solution of second and higher order complex linear differential equations has infinite order if the coefficients of them are entire functions and satisfy a certain conditions. Also we proved the above result for the special case of second order complex linear differential equation when the coefficients involved meromorphic functions and a nonconstant polynomial.*

Key words: complex differential equations; order; hyper order; entire function; meromorphic function.

1. INTRODUCTION

The order of growth of solutions of the equation

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0 \qquad (1)$$

is one of the aims in studying complex differential equations, where $A_0(z) \neq 0,\ A_k(z), 1 \leq k \leq n-1$ are entire functions. For the second order case

$$f'' + A(z)f' + B(z)f = 0 \qquad (2)$$

From [1, 2, 3], we know that every nontrivial solution of it has infinite order provided that

i) $\rho(A) < \rho(B)$; or ii) $\rho(B) < \rho(A) \leq \frac{1}{2}$ or iii) $A(z)$ is a polynomial and $B(z)$ is a transcendental entire function. For the case of polynomial coefficients, a classical result due to Wittich [4] says: if $A_k(z), 0 \leq k \leq n-1$ are entire functions, then all solutions of (1) are of infinite order if and only if all coefficients $A_k(z)$ are polynomials. Another result due to Frei [5] for (1) is: if $A_1(z), \ldots, A_j(z)$ are transcendental coefficients while $A_{j+1}(z), \ldots, A_{n-1}(z)$ are polynomials, then there can exist at most j linearly independent finite order solutions of (1).

In our work, we shall consider the solution growth of Eqs. (1), (2) and the following special case of Eq. (2)

$$f'' + h(z)e^{P(z)}f' + Q(z)f = 0 \qquad (3)$$

where $h(z), Q(z)$ are meromorphic functions and $P(z)$ is a nonconstant polynomial.

We assume that the reader is familiar with the fundamental results and standard notations in Nevanlinna theory, see [6, 7, 8] for more details.

G. Gunderson in 1988 proved the following for the Eq. (2):

Theorem 1.1 [2] Let $A(z), B(z) \neq 0$ be an entire functions such that for real constants $\alpha, \beta, \theta_1, \theta_2$ where $\alpha > 0, \beta > 0$ and $\theta_1 < \theta_2$ we have

$$|B(z)| \geq \exp\{(1 + o(1))\alpha\, |z|^{\beta}\}$$

and

$$|A(z)| \leq \exp\{o(1)|z|^{\beta}\}$$

as $z \longrightarrow \infty$ in $\theta_1 \leq arg z \leq \theta_2$. Then every non trivial solution f of Eq. (2) has infinite order.

Cai Feng Yi1, Xu-Qiang Liu and Hong Yan Xu in 2013 proved the following theorem:

Theorem 1.2 [9] Let $P(z)$ be a nonconstant polynomial with $degP = n$, let $h(z)$ be a meromorphic function with $\rho(h) < n$. Suppose that $Q(z)$ is a finite-order meromorphic function having an infinite deficient value, $Q(z)$ has only finitely many Borel directions $B_j\colon arg z = \theta_j, (j = 1,2, \ldots, q)$. Denote that $\Omega_j = \{z\colon \theta_j < arg z < \theta_{j+1}\}, j = 1,2, \ldots, q$. Suppose that there exists $\varphi_j(\theta_j < \varphi_j < \theta_{j+1})$ such that $\delta(P, \varphi_j) < 0$ for each angular domain Ω_j. Then every meromorphic solution $f \neq 0$ of equation (3) has infinite order with $\rho_2(f) \geq \rho(Q)$.

2. PRELIMINARIES

In what follows, we shall give some basic concepts related to the Nevanlinna theory of meromorphic functions.

Definition 2.1 [6, 7, 8] Let f be a meromorphic function. The order of growth and lower order of growth of f, denoted by $\rho(f)$ and $\mu(f)$ respectively are defined by

$$\rho(f) = \lim_{r\to\infty} sup \frac{log^{+}T(r,f)}{log\ r}$$

and

$$\mu(f) = \lim_{r\to\infty} inf \frac{log^{+}T(r,f)}{log\ r}$$

respectively. If f is entire, then $T(r,f)$ can be replaced with $log^{+}M(r,f)$, where $M(r,f) = \max_{|z|=r}|f(z)|$.

Definition 2.2 [5, 16] The hyper-order $\rho_2(f)$ of a meromorphic function f is defined by

$$\rho_2(f) = \lim_{r\to\infty} sup \frac{log^{+}log^{+}T(r,f)}{log\ r}$$

and

$$\rho_2(f) = \lim_{r\to\infty} sup \frac{log^{+}log^{+}log^{+}M(r,f)}{log\ r}$$

if f is entire.

Definition 2.3 [10] Let $f(z)$ be a meromorphic function in the complex plane with $\rho(f) = \rho,\ (0 < \rho \leq \infty)$. A ray $arg\ z = \theta$, $(0 \leq \theta < 2\pi)$ starting from the origin is called a Borel direction of order ρ of $f(z)$ if the following equality:

$$\lim_{r\to\infty} sup\ \frac{\log^+ n(\Omega(\theta-\varepsilon,\theta+\varepsilon;r),f=a)}{logr}=\rho$$

holds for any real number $\varepsilon > 0$ and every complex number $a \in \mathbb{C} \cup \{\infty\}$ with at most two exceptions, where $\Omega(\theta-\varepsilon,\theta+\varepsilon;r)=\{z: \theta-\varepsilon < argz < \theta+\varepsilon, |z|<r\}$.

3. SOME NEEDED LEMMAS

In this section, we shall introduce some results which will be useful in proving our results.

Lemma 3.1 [11] Let (f,Γ) be a pair contains a finite order transcendental, meromorphic function f and

$$\Gamma=\{(k_1,j_1),(k_2,j_2),\dots,(k_q,j_q)\}$$

denote a set of distinct integers order pairs satisfying $k_i > j_i \geq 0$, $i = 1,2,\dots,q$. Let $\varepsilon > 0$ be a constant. Then the following hold:

i) There is $E_1 \subset [0,2\pi)$ with zero linear measure, such that, when $\psi_0 \in [0,2\pi)\backslash E_1$, then a real constant $R_0=R_0(\psi_0)>1$ exists such that, for z with $arg\, z=\psi_0$ $|z|\geq R_0$, and for each $(k,j)\in\Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)} \tag{4}$$

ii) There is $E_2 \subseteq (1,\infty)$ with $m_l(E_2)<\infty$, such that, for each z with $|z|\notin E_2\cup[0,1]$ and, for each $(k,j)\in \Gamma$, we have (4).

iii) There is $E_3 \subset [0,\infty)$ with linear measure is finite, such that

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho+\varepsilon)} \tag{5}$$

for each z with $|z|\notin E_3$ and $(k,j)\in\Gamma$.

Lemma 3.2 [11] Let $(f\,,\Gamma)$ denote a pair that consists of a transcendental meromorphic function $f(z)$ and a finite set

$$\Gamma=\{(k_1,j_1),(k_2,j_2),\dots,(k_q,j_q)\}$$

of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ for $i = 1,2,\dots,q$. Let $\alpha>0$ and $\varepsilon>0$ be given real constants. Then the following three statements hold.

i) There exists a set $E_1 \subset [0,2\pi)$ that has linear measure zero, and there exists a constant $c>0$ that depends only on α and Γ such that if $\varphi_0 \in [0,2\pi)\backslash E_1$, then there is a constant $R_0=R_0(\varphi_0)>1$ such that for all z satisfying $arg\, z=\varphi_0$ and $|z|=r\geq R_0$, and for all $(k,j)\in\Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c\left(\frac{T(\alpha r,f)}{r}\ log^{\alpha} r \log T(\alpha r,f)\right)^{k-j} \tag{6}$$

In particular, if $f\,(z)$ has finite order $\rho(f\,)$, then (6) is replaced by:

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c|z|^{(k-j)(\rho(f)-1+\epsilon)} \tag{7}$$

ii) There exists a set $E_2 \subset [1,\infty)$ that has finite logarithmic measure, and there exists a constant $c>0$ that depends only on α and Γ such that for all z satisfying $|z|=r\notin E_2\cup[0,1]$ and for all $(k,j)\in\Gamma$, the inequality (6) holds. In particular, if $f\,(z)$ has finite order $\rho(f\,)$, then the inequality (7) holds.

iii) There exists a set $E_3 \subset [0,\infty)$ that has finite linear measure, and there exists a constant $c>0$ that depends only on α and Γ such that for all z satisfying $|z|=r\notin E_3$ and for all $(k,j)\in\Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c\ (T(\alpha r,f) r^{\varepsilon}\ log\ T(\alpha r,f))^{k-j} \tag{8}$$

In particular, if $f\,(z)$ has finite order $\rho(f\,)$, then (8) is replaced by

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq c|z|^{(k-j)(\rho(f)+\epsilon)} \tag{9}$$

Lemma 3.3 [12] Suppose that $g(z)=h(z)e^{P(z)}$, where $P(z)=(\alpha+i\beta)z^n+\cdots,(\alpha,\beta\in\mathbb{R})$ is a non-constant polynomial with $degP=n$, and $h(z)$ is a meromorphic function with $\rho(h)<n$. There exists a set $E_1\subset[0,2\pi)$ that has linear measure zero such that for all $\varphi\in[0,2\pi)\backslash E_1$, we have

i) If $\delta(P,\phi)<0$, where $\delta(P,\phi)=\alpha\, cosn\phi-\beta\, sinn\phi$, then there is a constant $R_0=R_0(\varphi_0)>0$ such that the inequality

$$\left|g(re^{i\varphi})\right| < \exp\{\tfrac{1}{2}\delta(P,\varphi)r^n\}$$

holds for $r>R_0$.

ii) If $\delta(P,\phi)>0$, then there is a constant $R'_0=R'_0(\varphi_0)>0$ such that the inequality

$$\left|g(re^{i\varphi})\right| < \exp\{\tfrac{1}{2}\delta(P,\varphi)r^n\}$$

holds for $r>R'_0$.

Lemma 3.4 [13] Let $g(z)$ be an entire function with $0<\mu(g)<\frac{1}{2}$ and let $A(z)$ be a meromorphic function with $\rho(A)<\infty$. If $A(z)$ has a finite deficient value a with deficiency $\delta=\delta(a,A)>0$, then for any given constant $\varepsilon>0$, there exists a sequence R_n with $R_n\longrightarrow\infty$, such that the following two inequalities

$$\left|g(R_n e^{i\varphi})\right| > \exp\{R_n^{\mu(B)-\varepsilon}\},\qquad \varphi\in[0,2\pi)$$

$$m(F_n)=:m\left\{\theta\in[0,2\pi)\left|log\left|A(R_ne^{i\theta})-a\right|\right.\geq\frac{-\delta}{4}T(R_n,A)\right\}\geq M_1>0$$

hold for all sufficiently large n, where M_1 is a constant depending only on $\rho(A),\mu(g)$ and δ.

Lemma 3.5 [14] Let $f(z)$ be an entire function with $0 \leq \mu(f) < 1/2$. Then for every $\alpha \in (\mu(f), 1)$, there exists a set $E \subset [0, \infty)$ such that $\overline{logdens}E \geq 1 - \frac{\mu(f)}{\alpha}$ where
$E = \{r \in [0, \infty): m(r) > M(r) cos\pi\alpha\}$,
$m(r) = \inf_{|z|=r} log|f(z)|$
and
$M(r) = \sup_{|z|=r} log|f(z)|$.

4. MAIN RESULTS

We generalize Theorem 1.1 to Eq. 1 as follows

Theorem 4.1 Suppose that $A_j(z), 0 \leq j \leq n-1$ $A_0(z) \neq 0$ are entire functions such that for real constants $\alpha, \beta, \theta_1, \theta_2$ where $\alpha > 0, \beta > 0$ and $\theta_1 < \theta_2$ we have

$$|A_0(z)| \geq \exp\{(1 + o(1))\alpha |z|^\beta\} \tag{10}$$

and

$$|A_j(z)| \leq \exp\{o(1)|z|^\beta\}, j = 1,2, \dots, n-1 \tag{11}$$

as $z \longrightarrow \infty$ in $\theta_1 \leq argz \leq \theta_2$. Then every non trivial solution f of (1) has infinite order.

Proof: Suppose that $f \neq 0$ is a solution of equation (1) where $\rho(f) < \infty$. Set $\rho = \rho(f)$. Then from Lemma 3.1 (i), there exists a real constant ψ_0 where $\theta_1 \leq \psi_0 \leq \theta_2$, such that

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq o(1)|z|^{j\rho}, 1 \leq j \leq n \tag{12}$$

as $z \longrightarrow \infty$ along $argz = \psi_0$. Then from (11) and (12) we obtain that

$$\begin{aligned}|A_0(z)| &\leq \left|\frac{f^n(z)}{f(z)}\right| + |A_{n-1}(z)| \left|\frac{f^{(n-1)}(z)}{f(z)}\right| + \cdots \\ &\quad + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right| \\ &\leq o(1)|z|^{n\mu} + |A_{n-1}(z)| o(1)|z|^{(n-1)\mu} + \cdots \\ &\quad + |A_1(z)| o(1)|z|^\mu\end{aligned}$$

as $z \longrightarrow \infty$ along $argz = \psi_0$, this contradicts (10) and (11).

We generalize Theorem 1.2 to be as follows in which the condition mentioned on Q is in place of that in Theorem 1.2:

Theorem 4.2 Let $P(z)$ be a nonconstant polynomial with $degP = n$, let $h(z)$ be a meromorphic function with $\rho(h) < n$. Let $Q(z)$ be a meromorphic function with $\mu(Q) < \frac{1}{2}$. Suppose that there exists an angular domain $\Omega(\theta, \epsilon, r) = \{z: \theta - \epsilon < argz < \theta + \epsilon, |z| < r\}$ such that $\delta(P, \varphi) < 0$ for some $\varphi \in \Omega(\theta, \epsilon, r)$. Then every meromorphic solution $f \neq 0$ of Eq. (3) has infinite order with $\rho_2(f) \geq \rho(Q)$.

Proof: Suppose that there exists a meromorphic solution $f \neq 0$ of Eq. (3) with $\rho(f) < \infty$. We shall seek a contradiction. From Eq. (3), we have the following inequality:

$$|Q(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + \left|h(z)e^{P(z)}\right| \left|\frac{f'(z)}{f(z)}\right| \tag{13}$$

By Lemma 3.2 (i), there exists a set $E_1 \subseteq [0, 2\pi)$ of measure zero and $R_0 > 0$ such that the following inequality:

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| < |z|^{2\rho(f)}, j = 1,2 \tag{14}$$

holds for all $z = |z|e^{i\varphi}$ with $\varphi \notin E_1$ and $|z| > R_0$. By Lemma 3.3, there exists a set $E_2 \subseteq [0, 2\pi)$ of measure zero and $R_0 > 0$ such that the following inequality:

$$\left|h(z)e^{P(z)}\right| < \exp\left\{\frac{1}{2}\delta(P, \varphi) r^n\right\} \tag{15}$$

holds for all $z = re^{i\varphi}$ satisfying $r > R'_0$ and $\varphi \in [0, 2\pi) \backslash E_2$. In the following, we shall consider two cases:

Case1: $0 < \mu(Q) < \frac{1}{2}$. Applying Lemma 3.4 to $Q(z)$, then for any given constants $\varepsilon > 0$, there exists a sequence r_m with $r_m \to \infty$ as $m \to \infty$ such that

$$\left|Q(r_m e^{i\varphi_m})\right| > \exp\left\{r_m^{\mu(Q)-\varepsilon}\right\} \tag{16}$$

holds for all sufficiently large m.

On the other hand, since there exists $\varphi \in \Omega(\theta, \varepsilon, r)$ such that $\delta(P, \varphi) < 0$ by the hypotheses, we can get an interval $[\theta'_1, \theta'_2] \in \Omega(\theta, \epsilon, r)$ such that (15) holds for all $z = re^{i\varphi}$ satisfying $r > R'_0$ and $\varphi \in [\theta'_1, \theta'_2] \backslash E_2$. Now, let $\xi = \frac{\theta'_2 - \theta'_1}{2}$. For each sufficiently large m, we can choose $\varphi_m \in [\theta'_1, \theta'_2] \backslash (E_1 . \cup E_2)$ such that (14), (15) and (16) hold for $z_m = r_m e^{i\varphi_n}$. Let $\eta = \frac{\rho(Q)}{2}$. Hence, from (13)-(16), we get

$$\exp\{r_m^\eta\} \leq r_m^{2\rho(f)} \left(1 + \exp\{\frac{1}{2}\delta(P, \varphi_m) r_m^n\}\right) \tag{17}$$

Obviously, when m is sufficiently large, this is a contradiction.

Case2: $\mu(Q) = 0$. By Lemma 3.5 there exists a set $E_3 \subseteq [0, \infty)$ with $\overline{logdens}E_3 = 1$ such that for all z satisfying $|z| = r \in E_3$, we have

$$log|Q(z)| > \frac{\sqrt{2}}{2} \log M(r, Q) \tag{18}$$

where $M(r, Q) = \max_{|z|=r}|Q(z)|$. It follows from Lemma 3.5 that there exists a sequence R_n such that (16) and (18) hold. From (13), (14) and (16), we can obtain (17). Hence, from (17) and (18), we get

$$M(r, Q)^{\frac{\sqrt{2}}{2}} \leq r_m^{2\rho(f)} \left(1 + \exp\{\frac{1}{2}\delta(P, \varphi_m) r_m^n\}\right)$$

For sufficiently large m, we get a contradiction from (17). Hence f must has infinite order.

Next, we shall prove that $\rho_2(f) \geq \rho(Q)$. By using Lemma 3.2 (i), there exist a set $E_4 \subseteq [0, 2\pi)$ of measure zero and two constants $B > 0$ and $R''_0 > 0$ such that for all z satisfying $|z| = r > R''_0$ and $argz \in E_4$, the following inequality holds:

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| < BT(2r, f)^4, \qquad j = 1,2$$

Hence, for each sufficiently large m, we can choose $\varphi_m \in [\theta'_1, \theta'_2] \setminus (E_2 \cup E_4)$ such that (15),(16) and (18) hold for $z_m = r_m e^{i\varphi'_m}$. From (13), (15), (16) and (18), we get

$$\exp\left\{r_m^{\rho(Q)-\eta}\right\} \leq BT(2r_m, f)^4 \left(1 + \exp\{\frac{1}{2}\delta(P, \varphi'_m) r_m^n\}\right)$$

Thus

$$\lim_{m\to\infty} \sup \frac{log^+ log^+ T(r_m, f)}{logr} \geq \rho(Q) - \eta$$

As η can be arbitrary small, we have $\rho_2(f) \geq \rho(Q)$. The proof of the Theorem is completed.

We generalize Theorem 1.1 by replacing the constants θ_1, θ_2 with two sets of real numbers $\{\varphi_k\}$ and $\{\theta_k\}, k = 1,2, \dots, n$ as follows:

Theorem 4.3 Let $A(z), B(z) \neq 0$ be an entire functions and let $\alpha > 0, \beta > 0$ be a real constants. Suppose that there exists two sets of real numbers $\{\varphi_k\}$ and $\{\theta_k\}, k = 1,2, \dots, n$ that satisfy $\varphi_1 < \theta_1 < \varphi_2 < \theta_2 < \cdots < \varphi_m < \theta_m < \varphi_{m+1} (\varphi_{m+1} = \varphi_1 + 2\pi)$ and

$$|B(z)| \geq \exp\{(1 + o(1))\alpha\, |z|^\beta\} \qquad (19)$$

and

$$|A(z)| \leq \exp\{o(1)|z|^\beta\} \qquad (20)$$

as $z \longrightarrow \infty$ in $\varphi_k \leq argz \leq \theta_k$. Then every nontrivial solution f of Eq. (2) has infinite order.

Proof: Suppose that $f \neq 0$ is a solution of Eq. (2) with $\rho = \rho(f) < \infty$. From Lemma 3.1 (i), there exists a set $E \subseteq [0,2\pi)$ with linear measure zero and a real constant φ_0 where $\varphi_1 \leq \varphi_0 \leq \theta_{n+1}$, such that

$$\left|\frac{f''(z)}{f(z)}\right| = o(1)|z|^{2\rho} \; and \; \left|\frac{f'(z)}{f(z)}\right| = o(1)|z|^\rho \qquad (21)$$

as $z \longrightarrow \infty$ along $argz = \varphi_0$. From Eq. (2) we obtain

$$|B(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + |A(z)| \left|\frac{f'(z)}{f(z)}\right| \qquad (22)$$

From (29), (21) and (22) we get a contradiction with Eq. (2). Therefore $\rho(f) = \infty$.

5. DISCUSSION AND CONCLUSION

In our work, the solution growth of the second and higher order complex linear differential equations with entire functions as coefficients and special type of second order have been discussed using the Nevanlinna theory of meromorphic functions. It is seen that the Nevanlinna theory is a powerful tool in studying these equations.

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