[illegible]

تاریخ/جمعه ۱۳۹۴/۱۲/۱۴

The multivariate Normal dist. $\boxed{MVN}$

The density fun. of anormal dist. $\underline{x}$ is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \qquad -\infty \le x \le \infty$$

The joint density of the indep. normal variables $x_1, \dots, x_p$ is:

$$f(x_1, \dots, x_p) = \frac{1}{(2\pi)^{p/2}\,\sigma_1 \cdots \sigma_p} e^{-\frac{1}{2}\sum_{i=1}^{p}\frac{(x-\mu)^2}{\sigma^2}}$$

Then $\underline{x}' = [x_1 \, \dots \, x_p]$, $\underline{\mu}' = [\mu_1 \dots \mu_p]$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ 0 & 0 & \sigma_3^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & & \sigma_p^2 \end{pmatrix}$$

Then the joint density can be given as:

$$f(\underline{x}) = \frac{1}{(2\pi)^{p/2}\,|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(\underline{x}-\underline{\mu})'\,\Sigma^{-1}(\underline{x}-\underline{\mu})\right]$$

This is the density fun. of $\underline{MVND.}$

where Σ : is $p*p$ matrix and p.d.

$f(\underline{x})$ is apositive for all $\underline{x}$

and $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\underline{x})\, dx_1 \dots dx_p = 1$

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$\therefore f(x)$ is adensity fun.

Note

Univariate

X متغير واحد

عدد المتغيرات واحد

μ متوسط.

σ^2 التباين لمتغير واحد.

التوزيع الطبيعي

$X \sim N(\mu, \sigma^2)$

لدينا الصيغة

$$-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2$$

الدالة لمتغير واحد.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

multivariate.

$$\underset{1\times p}{\underline{X}'} = [x_1 \ldots x_p]$$

p عدد المتغيرات

عدد المتغيرات = p.

$\underline{\mu}$ for Q. F $\quad \underline{\mu}' = [\mu_1 \ldots \mu_p]$ Q.F.

لدينا متجه متوسطات بعدد p. للصيغة التربيعية

$\underset{p\times p}{\Sigma}$ for Q. F.

مصفوفة التباينات والتباين المشترك. من درجة p*p للصيغة التربيعية.

$$\Sigma_{p\times p} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ & \sigma_2^2 & \cdots & \sigma_{2p} \\ & & \ddots & \\ & & & \sigma_p^2 \end{pmatrix}$$

$$\underset{p\times 1}{\underline{X}} \sim N_p(\underset{p\times 1}{\underline{\mu}}, \underset{p\times p}{\Sigma})$$

لدينا الصيغة التربيعية

Q. F.

$$Q = -\frac{1}{2}(\underline{X} - \underline{\mu})' \Sigma^{-1} (X - \mu)$$

الدالة لعدد من المتغيرات

$$f(\underline{X}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp -\frac{1}{2}\left[(\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\underline{\mu})\right]$$

2. Multivariate Normal Density Function

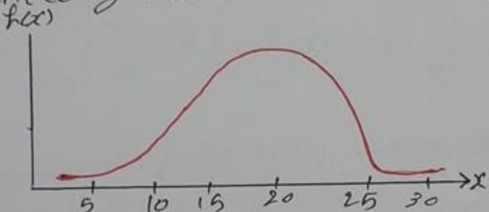
The multivariate normal density is an extension of the univariate normal distribution.

If X be a random variable, with mean μ and variance σ^2, is normally distribut, its density is given as:

The density of Normal dist. r.v. X is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

we say that $X \sim N(\mu, \sigma^2)$. This function is represented by the familiar bell-shaped curve.



The Normal Density Curve

Let $X_1, X_2, ---, X_p$ independent normal variable so that $E(X_i) = \mu_i$ and $var(X_i) = \sigma_i^2$ for $i = 1, 2, ---, p$.

The density function of the random vector $\underset{\sim}{X}' = (X_1, ..., X_p)$ is given as follows:

$$f(\underset{\sim}{x}) = \prod_{i=1}^{p} f(x_i) = \prod_{i=1}^{p} \frac{1}{\sqrt{2\pi}\,\sigma_i} \exp\left\{-\frac{(x_i-\mu_i)^2}{2\sigma_i^2}\right\}$$

$$f(\underset{\sim}{x}) = \frac{1}{(2\pi)^{p/2}\sigma_1 \cdots \sigma_p} \exp\left\{-\frac{1}{2}\sum_{i=1}^{p} \frac{(x_i-\mu_i)^2}{\sigma_i^2}\right\}$$

where $f(\underset{\sim}{x})$ or $\phi(X_1, ..., X_p) = \phi(\underset{\sim}{X})$ is the joint density of the p independent random vector $\underset{\sim}{X}' = (X_1, ..., X_p)$

Let $\underline{\mu}' = (\mu_1, \dots, \mu_p)$ and Know if we write $\underline{x} = (x_1, \dots, x_p)$

and $\Sigma_D = (\sigma_1^2, \dots, \sigma_p^2) = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ 0 & \cdots & & \sigma_p^2 \end{pmatrix}$

Then $\Sigma_D^{-1} = \left(\frac{1}{\sigma_1^2}, \dots, \frac{1}{\sigma_p^2}\right)$

And $|\Sigma_D| = \prod_{i=1}^{p} \sigma_i^2 = \sigma_1^2 * \sigma_2^2 * \dots * \sigma_p^2$.

$\Rightarrow |\Sigma_D|^{\frac{1}{2}} = \prod_{i=1}^{p} \sigma_i = \sigma_1 * \sigma_2 * \dots * \sigma_p$.

The term $\sum_{i=1}^{p} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2 = \sum_{i=1}^{p} (x_i - \mu_i) \frac{1}{\sigma_i^2} (x_i - \mu_i)$

$$= (x_1 - \mu_1) \frac{1}{\sigma_1^2} (x_1 - \mu_1) + \dots + (x_p - \mu_p) \frac{1}{\sigma_p^2} (x_p - \mu_p)$$

$$= \begin{pmatrix} x_1 - \mu_1 \\ \vdots \\ x_p - \mu_p \end{pmatrix}' \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2^2} & 0 & 0 \\ \vdots & & \ddots & \sigma_p^2 \\ 0 & \cdots & 0 & \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 \\ \vdots \\ x_p - \mu_p \end{pmatrix}$$

$$= (\underline{x} - \underline{\mu})' \Sigma_D^{-1} (\underline{x} - \underline{\mu})$$

Then the joint density can be given by:

Then the density function of $f(\underline{x})$ can be written as follows

$$f(\underline{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} (\underline{x} - \underline{\mu})' \Sigma_D^{-1} (\underline{x} - \underline{\mu})\right\}$$

ef. The random vector $\underset{p\times 1}{\underline{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$

is said to have a multivariate Normal dist.

with mean $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}$, and

variance - covariance matrix Σ, iff:

$$f(x_1, x_2, \dots, x_p, \Sigma) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})}$$

i.e $x \sim N_p(\underline{\mu}, \Sigma)$

where $\underset{p\times p}{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & & \sigma_{pp} \end{pmatrix}$, where Σ is symmetri $(\sigma_{ij} = \sigma_{ji})$ and (p. d. matr

and $\Sigma = E\underset{p\times 1}{(\underline{x}-\underline{\mu})}\underset{1\times p}{(\underline{x}-\underline{\mu})'}$

$\underset{p\times 1}{\underline{\mu}} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}$, $(\underline{x}-\underline{\mu}) = \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \\ \vdots \\ x_p-\mu_p \end{pmatrix}$

More generally, if $\underline{X}$ has a multivariate normal distribution with mean vector $\underline{\mu}$ and covariance matrix $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & & \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2 \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$

Then the density is given as follows:

$$f(\underline{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})\right\}$$

which it is denoted by $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$

Thats mean a p-dimension random vector $\underline{X}$ is said to have a p-variate normal distribution with mean vector $\underline{\mu}$ and covariance matrix Σ, where Σ be $(p*p)$ positive definite matrix $\Longrightarrow \det \Sigma = |\Sigma| > 0$.

we shall denote the density $f(\underline{x})$ as

$n(x | \underline{\mu}, \Sigma)$.

and $\int\int\cdots\int f(\underline{x})\, dx_1 \ldots dx_p = 1$

H.w Suppose that $Z_1, Z_2, \ldots, Z_p$ are independent random variables distributed $N(0,1)$.
Derive the density function of the multivariate normal distribution for a random vector $\underline{X}' = (x_1, x_2, \ldots, x_p)$.

Bi-variate Normal density func.

For the Bi-variate Normal dist.

(The case for $p = 2$ is especially important in statistical theory.) Hence

$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$, where Σ is the covariance matrix.

Consider the following cases.

① First case.

① $X \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$f(\underline{x}) = \phi(\underline{x}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp -\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})$$

$$\phi(\underline{x}) = f(\underline{x}) = \frac{1}{(2\pi)^{\frac{2}{2}} |\Sigma|^{1/2}} \exp -\frac{1}{2} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}' \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

$$= \frac{1}{2\pi \begin{vmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{vmatrix}^{1/2}} \exp -\frac{1}{2} \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

$$= \frac{1}{2\pi (\sigma_1^2 \sigma_2^2 - \rho^2 \sigma_1^2 \sigma_2^2)^{1/2}} \exp -\frac{1}{2} \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \frac{\begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{bmatrix}}{\sigma_1^2 \sigma_2^2 (1-\rho^2)} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

$$= \frac{1}{2\pi [\sigma_1^2 \sigma_2^2 (1-\rho^2)]^{1/2}} \exp -\frac{1}{2(1-\rho^2)} \begin{bmatrix} x_1 - \mu_1 & x_2 - \mu_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1 \sigma_2} \\ \frac{-\rho}{\sigma_2 \sigma_1} & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

$$= \frac{1}{2\pi \begin{vmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{vmatrix}^{1/2}} \exp -\frac{1}{2} \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= \frac{1}{2\pi (\sigma_1^2\sigma_2^2 - \rho^2 \sigma_1^2 \sigma_2^2)^{1/2}} \exp -\frac{1}{2} \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \frac{\begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{bmatrix}}{\sigma_1^2\sigma_2^2(1-\rho)} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= \frac{1}{2\pi [\sigma_1^2\sigma_2^2(1-\rho^2)]^{1/2}} \exp -\frac{1}{2(1-\rho^2)} \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= \frac{1}{2\pi \sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp -\frac{1}{2(1-\rho^2)} \begin{bmatrix} \frac{x_1-\mu_1}{\sigma_1^2} - \frac{\rho(x_2-\mu_2)}{\sigma_1\sigma_2} & -\frac{\rho(x_1-\mu_1)}{\sigma_1\sigma_2} + \frac{x_2-\mu_2}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= \frac{1}{2\pi \sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp -\frac{1}{2(1-\rho^2)} \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{\rho(x_2-\mu_2)(x_1-\mu_1)}{\sigma_1\sigma_2} - \frac{\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right]$$

$\phi(x_1,x_2) = f(x_1,x_2)$

$$\phi(\underline{x}) = f(\underline{x}) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp -\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right\}$$

$\therefore \quad (x_1, x_2) \sim N_2(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$.

The density fun. is for the Bivariat Normal dist.

Note that.

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}.$$

and $$\boxed{\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \Rightarrow \sigma_{12} = \rho \sigma_1 \sigma_2}$$

then
$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

and.
$$|\Sigma| = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2$$
$$\boxed{|\Sigma| = \sigma_1^2\sigma_2^2[1-\rho^2]}$$

So that
$$\Sigma^{-1} = \frac{1}{|\Sigma|}\,\mathrm{adj}(\Sigma).$$
$$\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$$
$$= \frac{1}{(1-\rho^2)} \begin{pmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix}$$

$$Q.f = (\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu})$$

$$= \frac{1}{1-\rho^2} \left[(\underline{x}_1 - \underline{\mu}_1) \quad (\underline{x}_2 - \underline{\mu}_2) \right] \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1 \sigma_2} \\ & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

$$= \frac{1}{1-\rho^2} \left[\frac{(x_1 - \mu_1)}{\sigma_1^2} - \frac{\rho (x_2 - \mu_2)}{\sigma_1 \sigma_2} \quad -\frac{\rho (x_1 - \mu_1)}{\sigma_1 \sigma_2} + \frac{(x_2 - \mu_2)}{\sigma_2^2} \right] \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}$$

$$= \frac{1}{1-\rho^2} \left\{ \frac{(x_1 - \mu_1)^2}{\sigma_1^2} - \frac{\rho (x_2 - \mu_2)(x_1 - \mu_1)}{\sigma_1 \sigma_2} - \frac{\rho (x_1 - \mu_1)(x_2 - \mu_2)}{\sigma_1 \sigma_2} + \frac{(x_2 - \mu_2)^2}{\sigma_2^2} \right\}$$

$$Q.f = \frac{1}{1-\rho^2} \left\{ \left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 - 2\rho \left(\frac{x_1 - \mu_1}{\sigma_1}\right)\left(\frac{x_2 - \mu_2}{\sigma_2}\right) + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2 \right\}$$

Then the MVN density.

$$f(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp -\frac{1}{2} \underbrace{\left[(x - \mu)' \Sigma^{-1} (x - \mu) \right]}_{Q.f.}$$

Then the MVN density:

$$f(x) = \frac{1}{(2\pi)^{p/2}\,|\Sigma|^{1/2}} \exp -\frac{1}{2} \underline{\left[(x-\mu)' \, \Sigma^{-1} (x-\mu) \right]} \quad Q.F.$$

This is defined to be in BVN dist. as:

$$f(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)} \left\{ \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho \left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right\} \right]$$

ويمكن كتابتها بالصيغة التالية.

$$\underset{\sim}{x} \sim N_2 (\underset{\sim}{\mu}, \Sigma)$$

where $\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ & \sigma_2^2 \end{pmatrix}$

نفرض ان $\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$

او تكتب بهذه الطريقة

$$(x_1, x_2) \sim N_2 (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

Bi-variate Normal Density Function

For the bivariate normal distribution with parameters $\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}. \text{ Since } \rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \rightarrow \sigma_{12} = \rho \sigma_1 \sigma_2$$

Then $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$ and

$$|\Sigma| = \sigma_1^2 \sigma_2^2 - \rho^2 \sigma_1^2 \sigma_2^2 = \sigma_1^2 \sigma_2^2 (1-\rho^2)$$

$$\therefore \Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$$

$$= \frac{1}{1-\rho^2} \begin{pmatrix} \frac{1}{\sigma_1^2} & -\rho/\sigma_1\sigma_2 \\ & \frac{1}{\sigma_2^2} \end{pmatrix}$$

Also $(\underline{X} - \underline{\mu})' \Sigma^{-1} (\underline{X} - \underline{\mu}) = \left[(x_1 - \mu_1) \quad (x_2 - \mu_2) \right]$

$$\frac{1}{1-\rho^2} \begin{pmatrix} \frac{1}{\sigma_1^2} & -\rho/\sigma_1\sigma_2 \\ & \frac{1}{\sigma_2^2} \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}$$

$$= \frac{1}{1-\rho^2} \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\frac{\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right]$$

Then the density function is defined as follows:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right) \right]$$

where ρ is the correlation between X_1 and X_2, this density is called the bivariate normal density function.

If the two random variables X_1 and X_2 are uncorrelated, so that $\rho = 0$, then

$$f(x_1, x_2) = f(x_1) * f(x_2)$$

Thats mean, X_1 and X_2 are independent.

From the density function of the bivariate normal distribution, it should be clear that, the quadratic form of the exponent in $f(x_1, x_2)$

$$-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1 \sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right] = Q$$

represent an ellipse with center at the point (μ_1, μ_2), called the centroid of the distribution.

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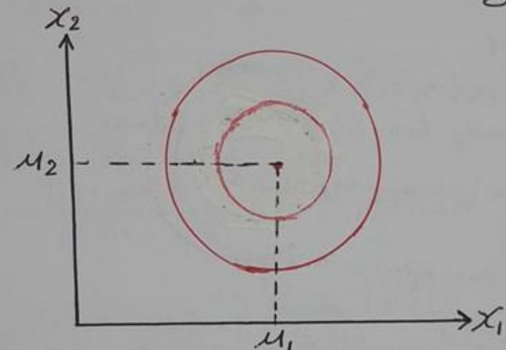
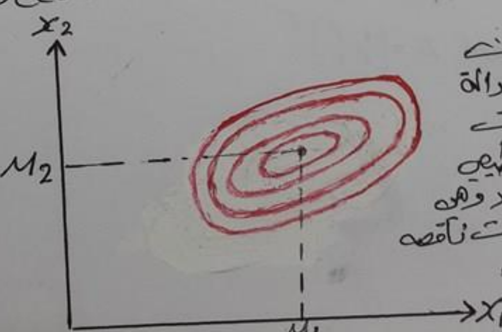
The shape of the ellipse in x_1x_2-plane depends on the value of the ratio of σ_{11} and σ_{22} $\left(\frac{\sigma_{11}}{\sigma_{22}}\right)$ and ρ.

(i) For $\sigma_{11} = \sigma_{22}$ and $\rho = 0$, the ellipse become a circle as in the following figure



(ii) For $\sigma_{11} = \sigma_{22}$ and $\rho > 0$, the orientation of the ellipse for various values of c becomes



تسمى الكنتورات ذات القيمة الدالة الثابتة لمتغيرات تتبع التوزيع الطبيعي الثنائي الابعاد وهي عبارة عن مجسمات ناقصة تتمركز عند $\underline{\mu}$

Constant-probability density ellipse bivariate normal

② <u>second case</u>

if $(X_1, X_2) \sim N_2(0, 0, \sigma_1^2, \sigma_2^2, \rho)$, يعني $\mu_1 = 0$, $\mu_2 = 0$

Then the density fun. is defined as follows:

$$\phi(\underline{x}) = f(\underline{x})$$

$$\phi(x_1, x_2) = f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp \frac{-1}{2(1-\rho^2)} \left[\frac{x_1^2}{\sigma_1^2} - \frac{2\rho x_1 x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right]$$

③ <u>third Case</u>. if

$(X_1, X_2) \sim N_2(0, 0, 1, 1, \rho)$

then the density fun. is defined as follows:

$$\phi(x_1, x_2) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp \frac{-1}{2(1-\rho^2)} \left\{x_1^2 - 2\rho x_1 x_2 + x_2^2\right\}$$

then we can also write it as:

$$\phi(x_1, x_2) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp \frac{-1}{2(1-\rho^2)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where

$$|\Sigma| = (1-\rho^2)$$

$$\Sigma^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

and $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

(4) Forth cases.

The standarized Bi-variate normal density with mean zero and unite variance.

Let $z_1 = \frac{x_1-\mu_1}{\sigma_1}$, $z_2 = \frac{x_2-\mu_2}{\sigma_2}$

Know.

$$x_1-\mu_1 = \sigma_1 z \quad \text{--- (1)}$$

$$x_2-\mu_2 = \sigma_2 z \quad \text{--- (2)}$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{vmatrix} = \begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{vmatrix} = \boxed{\sigma_1 \sigma_2}$$

$$g(z_1,z_2) = \frac{\sigma_1 \sigma_2}{2\pi \sigma_1 \sigma_2 (1-\rho^2)^{1/2}} \exp -\frac{1}{2(1-\rho^2)} \left[z_1^2 - 2\rho z_1 z_2 + z_2^2 \right]$$

$$g(z_1,z_2) = \frac{1}{2\pi (1-\rho^2)^{1/2}} \exp \frac{-1}{2} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' \frac{\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}}{(1-\rho^2)} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$g(z_1,z_2) = \frac{1}{2\pi (1-\rho^2)^{1/2}} \exp -\frac{1}{2} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$\therefore (z_1, z_2) \sim N_2(0, 0, 1, 1, \rho)$.

EX Let $\underline{X} \sim N_3(\underline{\mu}, \Sigma)$ and the pdf is

$$f(\underline{x}) = k \exp\{-Q/2\}, \text{ where}$$

$$Q = \tfrac{3}{2}x_1^2 + 2x_2^2 + x_3^2 - 3x_1x_2 + 2x_1x_3 - 2x_2x_3 + 10x_1 - 14x_2 + 8x_3 + 26$$

Find
(a) The mean vector $\underline{\mu}$ and var-cov matrix Σ.
(b) The value of the normalized constant k.
(c) The correlation matrix R.

Sol

$$Q = (\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu})$$

$$= \underline{x}' \Sigma^{-1} \underline{x} - 2\underline{\mu}' \Sigma^{-1} \underline{x} + \underline{\mu}' \Sigma^{-1} \underline{\mu}$$

where

$$\underline{x}' \Sigma^{-1} \underline{x} = (x_1 \;\; x_2 \;\; x_3) \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{3}{2} & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Longrightarrow \Sigma^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{3}{2} & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\therefore \Sigma = \frac{\operatorname{adj} \Sigma^{-1}}{|\Sigma|} = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix}$$

$$-2\underline{\mu}' \Sigma^{-1} \underline{x} = 10x_1 - 14x_2 + 8x_3$$

$$\therefore \underline{\mu}' \Sigma^{-1} \underline{x} = -5x_1 + 7x_2 - 4x_3$$

$$= (-5 \;\; 7 \;\; -4) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\underline{\mu}'\,\Sigma^{-1} = (-5 \quad 7 \quad -4)$$

$$\underline{\mu}' = (-5 \quad 7 \quad -4)\,\Sigma$$

$$= (-5 \quad 7 \quad -4)\begin{pmatrix} 4 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix}$$

$$= (2 \quad 4 \quad -2)$$

$$\therefore \underline{\mu} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$$

(b) $K = \dfrac{1}{(2\pi)^{P/2} |\Sigma|^{\frac{1}{2}}} = \dfrac{1}{(2\pi)^{3/2} |\Sigma|^{\frac{1}{2}}}$

$$|\Sigma| = \begin{vmatrix} 4 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 3 \end{vmatrix} = 4$$

$$\therefore K = \frac{1}{(15.759)(4)^{\frac{1}{2}}} = 0.0317$$

(c) $R = D^{-\frac{1}{2}}\,\Sigma\,D^{-\frac{1}{2}}$

$$= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 4 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0.707 & -0.577 \\ 0.707 & 1 & 0 \\ -0.577 & 0 & 1 \end{pmatrix}$$

Factorization of BVND.

From the BVND we can find the marginal dist. of X_1 & X_2 respectively

if $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

then

(1) The marginal dist. of X_1 ~~xxxx~~ is $N_1(\mu_1, \sigma_1^2)$

(2) - - - - - - X_2 - - - - $N_2(\mu_2, \sigma_2^2)$

(3) The conditional dist. of X_2 / X_1 is:

$$\boxed{\begin{aligned} E(X_2/X_1) &= \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(X_1 - \mu_1) \\ V(X_2/X_1) &= \sigma_2^2 (1-\rho^2) \end{aligned}}$$

(4) The conditional dist. of X_1 / X_2 is:

$$\boxed{\begin{aligned} E(X_1/X_2) &= \mu_1 + \rho \frac{\sigma_1}{\sigma_2}(X_2 - \mu_2) \\ V(X_1/X_2) &= \sigma_1^2 (1-\rho^2) \end{aligned}}$$

① To find $f(x_1)$. the marginal dist. of x_1..

$$f(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$$

لايجاد دالة x_1 فنكامل الدالة المشتركة $f(x_1,x_2)$ بالنسبة الى x_2.

$$f(x_2/x_1) = \frac{f(x_1, x_2)}{f(x_1)} \;\&\; f(x_1/x_2) = \frac{f(x_1, x_2)}{f(x_2)}$$

$$f(x_1, x_2) = f(x_2/x_1)\, f(x_1)$$

ولذلك يجب ان نعرف الدالة المشتركة $f(x_1,x_2)$ صحيحا

where

$$(x_1, x_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

$$\therefore \quad f(x_1, x_2) = \frac{1}{(2\pi)\sigma_1 \sigma_2 \sqrt{1-\rho^2}}\, e^{-\frac{1}{2} Q}$$

and

$$Q = \frac{1}{(1-\rho^2)} \left\{ \left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 - 2\rho \left(\frac{x_1 - \mu_1}{\sigma_1}\right)\left(\frac{x_2 - \mu_2}{\sigma_2}\right) + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2 \right\}$$

then $(1-\rho^2) Q = \{ \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad \}$ ... (*)

رقم المعادلة

and

$$Q = \frac{1}{(1-\rho^2)}\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}$$

then $(1-\rho^2)Q = \{ \quad " \quad " \quad " \quad " \quad \} \quad \ldots (*)$

رقم المعادلة

نضيف الى المعادلة $(*)$ $\mp \rho^2\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2$ فينتج

$$(1-\rho^2)Q = \left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 + \rho^2\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right\}$$

$$= \left\{\underbrace{\left[\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \rho^2\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right]}_{x_2^2 - 2a x_1 x_2 + a^2 x_1^2} + \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right\}$$

فرق ما بين مربعين {.....}

$$(1-\rho^2)Q = \left[\left(\frac{x_2-\mu_2}{\sigma_2}\right) - \rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\right]^2 + \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\{1-\rho^2\}$$

$$Q = \frac{1}{1-\rho^2}\left\{\left[\left(\frac{x_2-\mu_2}{\sigma_2}\right) - \rho\frac{(x_1-\mu_1)}{\sigma_1}\right]^2 + (1-\rho^2)\left[\frac{x_1-\mu_1}{\sigma_1}\right]^2\right\}$$

$$Q = \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \frac{1}{(1-\rho^2)}\left\{\frac{x_2-\mu_2}{\sigma_2} - \rho\,\frac{(x_1-\mu_1)}{\sigma_1}\right\}^2$$

$$Q = \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \frac{1}{\sigma_2^2(1-\rho^2)}\left\{x_2-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right\}^2$$

وبذلك فان

$$f(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$$

$$f(x_1) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left\{\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{1}{\sigma_2^2(1-\rho^2)}\left(x_2-\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)\right)^2\right\}}$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \cdot e^{-\frac{1}{2}\frac{1}{\sigma_2^2(1-\rho^2)}\left[x_2-\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)\right]^2}\, dx_2$$

$-\infty$

$$f(x_1) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left\{\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{1}{\sigma_2^2(1-\rho^2)}\left[\left(x_2-\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)\right)^2\right]\right\}}$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \cdot e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left[\left(x_2-\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)\right)\right]^2}\, dx_2$$

$$f(x_1) = \frac{1}{(2\pi)^{1/2}\sigma_1}\, e^{-\frac{1}{2}\frac{(x_1-\mu_1)^2}{\sigma_1^2}} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left(x_2-\overbrace{\boxed{\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)}}^{Ex_2}\right)^2} dx_2$$

This <u>means</u>

$$X_1 \sim N(\mu_1, \sigma_1^2)$$

$$X_2/X_1 \sim N\left[\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1),\ \sigma_2^2(1-\rho^2)\right]$$

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دائماً بالرمز المتوسط μ = mean

$x_2-\left(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)\right)$ x_2 عبر العلة المطروحة مرة من

اما الثاني فيمثل العلة المقسومة لان $\frac{x-\mu}{\sigma}$

وبذلك يكون $\sigma_2^2(1-\rho^2)$

H.3

Q1) let $(x,y) \sim N_2(0, 0, \sigma_1^2, \sigma_2^2, \rho)$

ص١٤ ، ١٧
الدفتر الابيض .

Show that the random variable

$$U = \frac{X}{\sigma_1} + \frac{y}{\sigma_2}$$

$$V = \frac{X}{\sigma_1} - \frac{y}{\sigma_2}$$

(a) are independent.

(b) $E(U)$, $E(V)$

(c) $Var(U)$, $Var(V)$

Q2) let $(x,y) \sim N_2(0, 0, \sigma_1^2, \sigma_2^2, \rho)$

Show that the variables.

$$U = \frac{X}{\sigma_1}$$

$$V = \frac{1}{\sqrt{1-\rho^2}} \left[\frac{y}{\sigma_2} - \rho \frac{X}{\sigma_1} \right]$$

(a) are standard Normal dist.

(b) are independent.

(c) Show that $U^2 + V^2 \sim \chi^2_{(2)}$

ص١٨ ، ص١٩ ، ص20

Ex: Derive the density fun. for.

(a) univariate normal dist.

(b) bivariate normal dist.

Ex. For the BVND. write the density fun. to

(a) $(X_1, X_2) \sim N_2(0, 0, \sigma_1^2, \sigma_2^2, \rho)$

(b) $(X_1, X_2) \sim N_2(0, 0, 1, 1, \rho)$

Sol

(a) $\underline{\mu} = \mu_1$, $\Sigma = \sigma_{11}^2 = \sigma_1^2$

$|\Sigma| = \sigma_1^2$ نفسه

$P = 1$.

then

$$f(x, \mu, \sigma^2) = \frac{1}{(2\pi)^{1/2} (\sigma_1^2)^{1/2}} e^{-\frac{1}{2}(x-\mu_1)'(\sigma_1^2)^{-1}(x-\mu_1)}$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)'(x-\mu_1)}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$x \sim N(\mu, \sigma^2)$

Univariate normal dist.

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(b) Bivariate N.d. (BND)

$$\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$$|\Sigma| = \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}$$

$$\boxed{|\Sigma| = \sigma_{11}\sigma_{22} - \sigma_{12}^2}$$

But we know that $\rho = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2}$

$$\Rightarrow \boxed{\sigma_{12} = \rho\sigma_1\sigma_2}$$

then

$$|\Sigma| = \sigma_{11}\sigma_{22} - (\rho\sigma_1\sigma_2)^2$$

but $\sigma_{11} = \sigma_1^2$

$\sigma_{22} = \sigma_2^2$

$$|\Sigma| = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2$$

$$\boxed{|\Sigma| = \sigma_1^2\sigma_2^2(1-\rho^2)}$$

or $\boxed{|\Sigma| = \sigma_{11}\sigma_{22}(1-\rho^2)}$

or

$$|\Sigma| = \sigma_{11}\sigma_{22} - \rho^2\sigma_{11}\sigma_{22}$$

$$\boxed{|\Sigma| = \sigma_{11}\sigma_{22}(1-\rho^2)}$$

$$\therefore \Sigma^{-1} = \frac{1}{|\Sigma|}\,\mathrm{adj}(\Sigma)$$

$$= \frac{1}{\sigma_{11}\sigma_{22}(1-\rho^2)} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$$

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$$\Sigma^{-1} = \frac{1}{(1-\rho^2)} \begin{pmatrix} \frac{\sigma_{22}}{\sigma_{11}\sigma_{22}} & -\frac{\sigma_{12}}{\sigma_{11}\sigma_{22}} \\ \frac{-\sigma_{12}}{\sigma_{11}\sigma_{22}} & \frac{\sigma_{11}}{\sigma_{11}\sigma_{22}} \end{pmatrix}$$

But $\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$

then

$$\Sigma^{-1} = \frac{1}{1-\rho^2} \begin{pmatrix} \frac{1}{\sigma_{11}} & \frac{-\sigma_{12}}{\sigma_1\sigma_1\sigma_2\sigma_2} \\ \frac{-\sigma_{12}}{\sigma_1\sigma_1\sigma_2\sigma_2} & \frac{1}{\sigma_{22}} \end{pmatrix}$$

$$= \frac{1}{1-\rho^2} \begin{pmatrix} \frac{1}{\sigma_{11}} & -\frac{\sigma_{12}}{\sigma_1\sigma_2\,\sigma_1\sigma_2} \\ \frac{-\sigma_{12}}{\sigma_1\sigma_2\,\sigma_1\sigma_2} & \frac{1}{\sigma_{22}} \end{pmatrix}$$

ρ ρ

$$\boxed{\Sigma^{-1} = \frac{-1}{1-\rho^2} \begin{pmatrix} \frac{1}{\sigma_{11}} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_{22}} \end{pmatrix}}$$

Know we found the Q.f. where

$$Q.F. = (\underline{x}-\underline{\mu})' \Sigma^{-1} (x-\mu)$$

then $(\underline{X}-\underline{\mu}) = \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$, $(\underline{x}-\underline{\mu})' = \left((\underline{x}_1-\underline{\mu}_1) \;\; (x_2-\mu_1) \right)$

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$$\Sigma^{-1} = \frac{1}{(1-\rho^2)} \begin{pmatrix} \frac{\cancel{\sigma_{22}}}{\sigma_{11}\cancel{\sigma_{22}}} & -\frac{\sigma_{12}}{\sigma_{11}\sigma_{22}} \\ \frac{-\sigma_{12}}{\sigma_{11}\sigma_{22}} & \frac{\cancel{\sigma_{11}}}{\sigma_{22}\cancel{\sigma_{11}}} \end{pmatrix}$$

But $\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$

then

$$\Sigma^{-1} = \frac{1}{1-\rho^2} \begin{pmatrix} \frac{1}{\sigma_{11}} & \frac{-\sigma_{12}}{\sigma_1\sigma_1\sigma_2\sigma_2} \\ \frac{-\sigma_{12}}{\sigma_1\sigma_1\sigma_2\sigma_2} & \frac{1}{\sigma_{22}} \end{pmatrix}$$

$$= \frac{1}{1-\rho^2} \begin{pmatrix} \frac{1}{\sigma_{11}} & -\frac{1}{\sigma_1\sigma_2}\underbrace{\frac{\sigma_{12}}{\sigma_1\sigma_2}}_{\rho} \\ -\frac{1}{\sigma_1\sigma_2}\underbrace{\frac{\sigma_{12}}{\sigma_1\sigma_2}}_{\rho} & \frac{1}{\sigma_{22}} \end{pmatrix}$$

$$\boxed{\Sigma^{-1} = \frac{-1}{1-\rho^2} \begin{pmatrix} \frac{1}{\sigma_{11}} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_{22}} \end{pmatrix}}$$

$$\frac{1}{1-\rho^2}\begin{pmatrix} & \\ \frac{-\sigma_{12}}{\sigma_1\sigma_2\,\sigma_1\sigma_2} & \frac{1}{\sigma_{22}}\end{pmatrix}$$

ρ

$$\boxed{\Sigma^{-1} = \frac{-1}{1-\rho^2}\begin{pmatrix} \frac{1}{\sigma_{11}} & -\frac{\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_{22}} \end{pmatrix}}$$

Know we found the Q.f. where

$$Q.f. = (\underline{x}-\underline{\mu})'\,\Sigma^{-1}\,(x-\mu)$$

then $(\underline{x}-\underline{\mu}) = \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$, $(\underline{x}-\underline{\mu})' = \left((\underline{x}_1-\underline{\mu}_1)\ (x_2-\mu_2)\right)$

Then we have.

$$f(\underline{x},\underline{\mu},\Sigma)=\frac{1}{(2\pi)^{p/2}\,|\Sigma|^{1/2}}\,e^{-\frac{1}{2}(\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})}$$

$$f(x_1,x_2,\mu_1,\mu_2,\sigma_{11},\sigma_{22},\rho)=\frac{1}{(2\pi)^{p/2}\left(\sigma_{11}\sigma_{22}(1-\rho^2)\right)^{1/2}}\ \exp$$

$$-\frac{1}{2(1-\rho^2)}\left\{\left[(x_1-\mu_1)\ \ (x_2-\mu_2)\right]\begin{bmatrix}\frac{1}{\sigma_{11}} & \frac{-\rho}{\sigma_1\sigma_2}\\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_{22}}\end{bmatrix}\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}\right\}$$

$$=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}}\exp\left[\frac{-1}{2(1-\rho^2)}\left\{\left[\frac{(x_1-\mu_1)}{\sigma_{11}}-\frac{\rho(x_2-\mu_2)}{\sigma_1\sigma_2}\quad -\frac{\rho(x_1-\mu_1)}{\sigma_1\sigma_2}+\frac{(x_2-\mu_2)}{\sigma_{22}}\right]\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}\right\}\right]$$

$$=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left\{\frac{(x_1-\mu_1)^2}{\sigma_{11}}-\frac{\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}-\frac{\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(x_2-\mu_2)^2}{\sigma_{22}}\right\}\right.$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} \exp\left[\frac{-1}{2(1-\rho^2)} \begin{bmatrix} \frac{(x_1-\mu_1)}{\sigma_{11}} - \frac{\rho(x_2-\mu_2)}{\sigma_1\sigma_2} & -\frac{\rho(x_1-\mu_1)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)}{\sigma_{22}} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \right]$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left\{ \frac{(x_1-\mu_1)^2}{\sigma_{11}} - \frac{\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} - \frac{\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_{22}} \right\}\right\}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left\{ \frac{(x_1-\mu_1)^2}{\sigma_{11}} - 2\,\frac{\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_{22}} \right\}\right\}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right\}\right\}$$

Know if $\rho=0$ then x_1 and x_2 are indep.

if $\rho>0$ then x_1 & x_2 positive corelated.

107 if $\rho<0$ then x_1 & x_2 are negative corelated

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why we study the normal multivariat dist.

(1) The marginal dist. and conditional dist. derived from multivariate normal dist. are also Normal dist.'s.

(2) Linear Combinations of normal variable are also normaly dist.

3. Some of the properties of the Multivariat Normal distribution

we list some of the properties of a random vector $\underline{X}$ from multivariate Normal distribution $N_p(\underline{\mu}, \Sigma)$

Theorem 1 A standarized vector $\underline{Z} = \Sigma^{-\frac{1}{2}}(\underline{X} - \underline{\mu})$ where $\Sigma^{-\frac{1}{2}}$ is the symmetric square root matr of Σ such that $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$.

The standard vector of random variables has all mean equal to zero's, all variances equal to one. It follows that $\underline{Z}$ is multivariate normal:

If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, then $\underline{Z} \sim N_p(\underline{0}, I)$.

Theorem 2 Let $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, then

$$\underline{Y} = C\underline{X} \sim N_p(C\underline{\mu}, C\Sigma C')$$

for C nonsingular matrix.

proof. The density of $\underline{Y}$ is obtained from density of $\underline{X}$, $n(\underline{x} | \underline{\mu}, \Sigma)$ by replacing $\underline{X}$ by $C^{-1}\underline{Y}$ and multiplying by the jacobian of the transformation

$$|J| = \text{Mod}\left|\frac{d\underline{X}}{d\underline{Y}}\right| = \text{Mod}|C^{-1}|$$

$$= \frac{1}{\text{mod}|C|} = \sqrt{\frac{1}{|C|^2}}$$

$$= \sqrt{\frac{1}{|C||C'|}} = \sqrt{\frac{|\Sigma|}{|C||\Sigma||C'|}}$$

$$|J| = \frac{|\Sigma|^{\frac{1}{2}}}{|C\Sigma C'|^{\frac{1}{2}}}$$

The quadratic form in the exponent of $n(\underline{x}|\underline{\mu},\Sigma$
is $Q = (\underline{x}-\underline{\mu})'\Sigma^{-1}(\underline{x}-\underline{\mu})$

$$= (C^{-1}\underline{y}-\underline{\mu})'\Sigma^{-1}(C^{-1}\underline{y}-\underline{\mu})$$

$$= (C^{-1}\underline{y}-C^{-1}C\underline{\mu})'\Sigma^{-1}(C^{-1}\underline{y}-C^{-1}C\underline{\mu})$$

$$= (\underline{y}-C\underline{\mu})'(C^{-1})'\Sigma^{-1}C^{-1}(\underline{y}-C\underline{\mu})$$

$$= (\underline{y}-C\underline{\mu})'(C\Sigma C')^{-1}(\underline{y}-C\underline{\mu})$$

since $(C^{-1})' = (C')^{-1}$ by virtue of transpositi
of $CC^{-1} = I$.

Thus, the density of $\underline{y}$ is

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$$f(\underline{y}) = n(C^{-1}\underline{y}|\underline{\mu},\Sigma) * \text{Mod}|C^{-1}|$$

$$= \frac{|\Sigma|^{\frac{1}{2}}}{|C\Sigma C'|^{\frac{1}{2}}} * \frac{1}{(2\pi)^{p/2}|\Sigma|^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}(\underline{y}-C\underline{\mu})'(C\Sigma C')^{-1}(\underline{y}-C\underline{\mu})\}$$

$$= \frac{1}{(2\pi)^{p/2}|C\Sigma C'|^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}(\underline{y}-C\underline{\mu})'(C\Sigma C')^{-1}(\underline{y}-C\underline{\mu})\}$$

$$= n(\underline{y}|C\underline{\mu}, C\Sigma C')$$

$$\therefore\ \underline{y} \sim N_p(C\underline{\mu}, C\Sigma C')$$