Th. 2:

let X be a p- component vector. dist. as. عنصر

$N_p(\underline{\mu}, \Sigma)$. then

$\underline{Y} = C\,\underline{X}$ is adistributed according to

$\underline{Y} = C\underline{X} \sim N_p(C\underline{\mu}, C\Sigma C')$

for C non singular.

proof: The density of $\underline{Y}$ is obtained from the density of $\underline{X}$.

by replacing x by $C^{-1}y$ and multiplying by the

jacobian of the transformation

jacobian of the transformation

لدينا $\underline{X}$ توزيعه معروف

ونريد توزيع $\underline{Y}$ من العلاقة مع X .

لذلك نقوم بعملية التحويل للمتغيرات.

حيث

$$f(y) = g(x = u^{-1}(y)) \cdot |J|$$

ولابد من :

ذكر

1) دالة التوزيع لـ X

2) ذكر العلاقة وتحويلها لـ X ..

3) ذكر [illegible] التحويل.

$$\underline{X} \sim N_p(\underline{\mu}, \Sigma)$$

$$f(\underline{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})\right\}$$

$$\underline{Y} = C\underline{X} \qquad , \quad |C| \neq 0$$

C nonsingular

$$\underline{X} = C^{-1}\underline{Y}$$

$$f(y) = f(C^{-1}Y) \cdot |J|$$

Th. 2:

let X be a p-component vector. dist. as. عنصر

$N_p(\underline{\mu}, \Sigma)$. then

$\underline{Y} = C\,\underline{X}$ is a distributed according to

$$\underline{Y} = C\underline{X} \sim N_p(C\underline{\mu}, C \Sigma C')$$

for C non singular.

proof: The density of $\underline{Y}$ is obtained from the density of $\underline{X}$. by replacing x by $C^{-1}y$ and multiplying by the jacobian of the transformation by replacing x by $c^{-1}y$ and multiplying by the jacobian of the transformation

لدينا $\underset{\sim}{X}$ توزيعه معروف.

ونريد توزيع $\underset{\sim}{Y}$ من العلاقة مع X .

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حيث

$$f(y) = g(x = u^{-1}(y)) \cdot |J|$$

وكالآتي :

$$\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$$

$$f(\underset{\sim}{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} (\underset{\sim}{x} - \underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu})\right\}$$

ذكرت
① دالة التوزيع لـ X

$$\underset{\sim}{Y} = C \underset{\sim}{X} \quad , \quad |C| \neq 0$$

C nonsingular

② ذكرت العلاقة.
وحولتها الى X ..

$$\underset{\sim}{X} = C^{-1} \underset{\sim}{Y}$$

③ ذكرت حساب التحويل.

$$f(y) = f(C^{-1}y) \cdot |J|$$

مطلق الجاكوبين

$$f(y) = f(c^{-1}y) \cdot |J|.$$

نجد ال jacobian

$$J = \left|\frac{\partial x}{\partial y}\right|$$

هو محدد المشتقة

where $J = \text{abs} \left|\frac{\partial x}{\partial y}\right|$

المطلق

محدد

الجاكوبين

$$|J| \underset{\text{mod}}{=} \left|\frac{\partial x}{\partial y}\right| = |c^{-1}|.$$

Know:

$$J = |c^{-1}|$$

$$= \frac{1}{|c|}$$

we have

$$\boxed{\begin{aligned} cc^{-1} &= I \\ |cc^{-1}| &= |I| \\ |c| \cdot |c^{-1}| &= 1 \\ |c^{-1}| &= \frac{1}{|c|} \end{aligned}}$$

$$= \frac{1}{|C|}$$

$$\boxed{|C^{-1}| = \frac{1}{|C|}}$$

$$= \sqrt{\frac{1}{|C|^2}}$$

نربع ونجذر

$$\rightarrow \quad |C|^2 = |CC'| = |C| \cdot |C'|$$

because.

$$|AB| = |A| \cdot |B|.$$

نضرب ونقسم على $|\Sigma|^{1/2}$ [illegible]

$$= \frac{|\Sigma|^{1/2}}{|\Sigma|^{1/2}} \cdot \frac{\sqrt{1}}{\sqrt{|C| \cdot |C'|}}$$

$$|A| = |A'|$$

$$= \sqrt{\frac{|\Sigma|}{|C| \cdot |\Sigma| \, |C'|}}$$

$$\rightarrow \quad |ABC| = |A| \cdot |B| \cdot |C|.$$

$$= \sqrt{\frac{|\Sigma|}{|C \Sigma C'|}}$$

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mod
abs.
الحد المطلق

$$|J| = \frac{|\Sigma|^{1/2}}{|C \Sigma C'|^{1/2}}$$

let the Quadratic form. in the $n(\underline{x} \mid \underline{\mu}, \Sigma)$

$$Q = (\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu})$$

we have $\quad Y = cX$

then $\quad X = c^{-1} Y.$

$$E X = c^{-1} E Y$$

$$= c^{-1} E(cX)$$

$$= c^{-1} c \, EX$$

$$\mu = \boxed{c^{-1} c \, \mu}$$

then.

$$Q = (c^{-1} y - c^{-1} c \mu)' \Sigma^{-1} (c^{-1} y - c^{-1} c \mu)$$

$$Q = (c^{-1}y - c^{-1}c\mu)'\,\Sigma^{-1}\,(c^{-1}y - c^{-1}c\mu)$$

$$= (c^{-1}(y - c\mu))'\,\Sigma^{-1}\,c^{-1}(y - c\mu)$$

$$= (y - c\mu)'(c^{-1})'\,\Sigma^{-1}\,c^{-1}(y - c\mu)$$

$$(c^{-1})' = (c')^{-1}$$

$$Q = (y - c\mu)'(c')^{-1}\,\Sigma^{-1}\,c^{-1}(y - c\mu)$$

$$\boxed{Q. = (y - c\mu)'\,(c\Sigma c')^{-1}\,(y - c\mu)}$$

The density of $\underline{y}.$ is

$$f(\underline{y}) = n(\underbrace{c^{-1}y}_{x=} \mid \underline{\mu}, \Sigma) * (\overbrace{\text{mod } |c^{-1}|}^{|J|}).$$

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$$f(y) = \frac{1}{(2\pi)^{p/2}\cancel{|\Sigma|^{1/2}}} \exp\left\{-\frac{1}{2}(\underline{y} - c\underline{\mu})'(c\Sigma c')^{-1}(\underline{y} - c\underline{\mu})\right\} * \frac{\cancel{|\Sigma|^{1/2}}}{|c\Sigma c'|^{1/2}}$$

$$\therefore\ f(y) = \frac{1}{(2\pi)^{p/2}\,|c\Sigma c'|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{y} - c\underline{\mu})'(c\Sigma c')^{-1}(\underline{y} - c\underline{\mu})\right\}$$

$$= n(\underline{y} \mid c\underline{\mu}, c\Sigma c')$$

المتغير   المعلمة الأولى   المعلمة الثانية

Theorem 2

نظرية مهمة

نفس الاشتقاقات
السابقة ولكن مختصره

let $X \sim N_p(\underline{\mu}, \Sigma)$

then $\underset{p\times 1}{Y} = \underset{p\times p}{C}\, \underset{p\times 1}{\underline{X}} \sim N_p(C\underline{\mu},\ C \Sigma C')$

where C is non singular matrix.

proof we have

$$\underline{Y} = C\, \underline{X}$$

$$\Rightarrow \underline{X} = C^{-1}\, \underline{Y}$$

تحويل .

$x = u(y)$

$y = u^{-1}(x)$

$f(y) = g(x) \cdot |J|$

$$f(\underline{y}) = |J| \cdot f(\underline{x}, \underline{\mu}, \Sigma)$$

$$f(\underline{y}) = |J| \cdot f(C^{-1}\underline{y}, \mu, \Sigma)$$

where

$$EX = C^{-1}\, EY$$

$$= C^{-1}\, E(CY) = C^{-1} C(EY) = C^{-1} C \mu = I\mu = \boxed{\mu}$$

where

$$Ex = c^{-1} EY$$
$$= c^{-1} E(cY) = c^{-1} c(EY) = c^{-1} c \mu = I \mu = \boxed{\mu}$$

$$|J| = \operatorname{mod}\left|\frac{\partial x}{\partial y}\right| = \operatorname{mod}|c^{-1}| = \operatorname{mod}\frac{1}{|c|}$$

القيمة المطلقة للمحدد.

محدد:

$$= \frac{1}{\sqrt{|c|^2}} = \frac{|\Sigma|^{1/2}}{|\Sigma|^{1/2}} \cdot \frac{1}{\sqrt{|c| \cdot |\Sigma|}}$$

$$= \frac{|\Sigma|^{1/2}}{\sqrt{|c|\,|\Sigma|\,|c'|}}$$

$$= \frac{|\Sigma|^{1/2}}{\sqrt{|c \Sigma c'|}}$$

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$$= \frac{|\Sigma|^{1/2}}{|c \Sigma c'|^{1/2}}$$

$$\therefore f(y) = (\underline{y}, c\underline{\mu}, \underline{c} \Sigma \underline{c}') = \frac{1}{(2\pi)^{1/2} |\Sigma|^{1/2}} \cdot |J| \cdot \exp\left\{-\frac{1}{2}[c^{-1}\underline{y} - c^{-1}c\underline{\mu}]\, \Sigma^{-1} (c^{-1}\underline{y} - c^{-1}c\underline{\mu})\right\}$$

<u>Note</u>

$$\left[C^{-1}(\underline{Y}-C\underline{\mu})\right]'\,\Sigma^{-1}\,C^{-1}(\underline{Y}-C\underline{\mu})$$

$$(\underline{Y}-C\underline{\mu})'\,C^{-1'}\,\Sigma^{-1}\,C^{-1}(\underline{Y}-C\underline{\mu})$$

$$(Y-C\mu)'\,(C\Sigma C')^{-1}(\underline{Y}-C\underline{\mu})$$

$$\therefore\; f(\underline{Y}) = \frac{|\Sigma|^{1/2}}{(2\pi)^{p/2}\,|\Sigma|^{1/2}\,|C\Sigma C'|^{1/2}}\exp\{-\tfrac{1}{2}(Y-C\mu)'(C\Sigma C')^{-1}(\underline{Y}-C$$

$$Y \sim N_p(C\underline{\mu},\ \underline{C}\,\Sigma\,\underline{C}')$$

<u>Th.2</u> if $X \sim MVN(\underline{\mu},\Sigma)$

The ~~moving~~ moment g.f. of X is:

$$M_x(t) = e^{\underline{t}'\underline{\mu} + \frac{1}{2}\underline{t}'\Sigma\underline{t}}.$$

الفائدة منه
لايجاد التوزيعات.
الاشتقاق عليه

Ex. let $X \sim N_3 \left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \right]$.

suppose that.

$\underset{\sim}{Y} = C \underset{\sim}{X}$ where $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Find the dist. of Y.

from th.2. we have

$Y \sim N_p(C\mu, C\Sigma C')$, $p = 3$.

then

$$C\mu = \underset{3\times3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{3\times1}{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$C\Sigma C' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C\Sigma C' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Then $X \sim N_3(\underset{\sim}{\mu}, \Sigma)$

so that $Y \sim N_3(C\underset{\sim}{\mu}, C\Sigma C')$ Th.2.

then $Y \sim N_3\left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right)$

من النظرية 2

From th. 2 if $\underline{x} \sim N_p(\underline{\mu}, \Sigma)$ and c is any $m \times p$ matrix with rank $m \leq p$ the new m-components random vector $\underline{y} = c\,\underline{x}$ is dist. acorrding to

$$\underline{y} \sim N_m(c\underline{\mu},\ c\Sigma c')$$

From this theorem we have

1) if $m=1$ the scalar variate

$$c'x \sim N_1(c'\mu,\ c\Sigma c')$$

it is univeriate Normal dist.

2) The joint dist. of any set. of element of $\underline{x}$ is multinormal with mean vector and v-c matrix given by the appropriate elements of $\underline{\mu}$ and Σ.

Hint.

Q) let $\underline{X} \sim N_2(\underline{\mu}, \Sigma)$

$$\underline{\mu} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 16 & -4 \\ & 12 \end{pmatrix}$$

① Find the dist. of $\underline{Y}$.?

② write the density fun. of $\underline{Y}$?

where

$$Y_1 = X_1 + 2X_2$$
$$Y_2 = X_1 - 3X_2$$

Q) let $X \sim N_3\left[\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 15 & 1 & -1 \\ & 6 & 2 \\ & & 4 \end{pmatrix}\right]$

suppose that $\underline{Y} = C\underline{X}$

① Find the dist. of $\underline{Y}$. ? where [illegible]

② suppose that

$$\cancel{Y}_1 = X_1 - 2X_2 + X_3$$
$$Y_2 = X_1 + X_2 + X_3$$

Find the dist of $\underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

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3) Let $\underline{Z} = \underline{c}'\underline{X}$

where $c' = (1 \quad -1 \quad 1)$

Find dist. of $\underline{Z}$?

4) find the joint. dist for X_1, X_3

Remarks

① If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$. If $\underline{a}$ is a vector of constant, the linear function $Y = a_1 x_1 + \cdots + a_p x_p$ $= \underline{a}' \underline{X}$ is univariate normal:

If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, then $Y = \underline{a}'\underline{X} \sim N(\underline{a}'\underline{\mu}, \underline{a}'\Sigma\underline{a})$

② If A is constant $(q \times p)$ matrix of rank q, where $q \le p$, and $\underline{a}$ a vector of constant of order q

Then:

If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, then

$$\underline{Y} = A\underline{X} + \underline{a} \sim N_q(A\underline{\mu} + \underline{a}, A\Sigma A')$$

Ex Let $\underline{X} \sim N_3(\underline{\mu}, \Sigma)$ with $\underline{\mu} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

and $\Sigma = \begin{pmatrix} 6 & 1 & -2 \\ & 13 & 4 \\ \cdot & & 4 \end{pmatrix}$

Find

ⓐ The distribution of $y = 2x_1 - x_2 + 3x_3$

ⓑ The distribution of $y_1 = x_1 + x_2 + x_3$, $y_2 = x_1 - x_2 + 2x_3$

ⓒ The distribution of $y_1 = x_1 + x_2 + 2$, $y_2 = 2x_1 + 3$

Sol

(a) $y = 2x_1 - x_2 + 3x_3 = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{a}' \underline{X}$

$E(y) = E(\underline{a}'\underline{X}) = \underline{a}' E(\underline{X}) = \underline{a}' \underline{\mu} = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

$= 17$

$var(y) = var(\underline{a}'\underline{X}) = \underline{a}' \Sigma \underline{a}$

$= \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 6 & 1 & -2 \\ & 13 & 4 \\ & & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 21$

$\therefore y \sim N(17, 21)$

$h(y) = \dfrac{1}{\sqrt{2\pi}\sqrt{21}} e^{-(x-17)^2/2(21)}$

(b) Let $\underline{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 + 2x_3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C\underline{X}$

$E(\underline{Y}) = C E(\underline{X}) = C\underline{\mu} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 10 \end{pmatrix}$

$var(\underline{Y}) = var(C\underline{X}) = C \Sigma C'$

120 $= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 6 & 1 & -2 \\ & 13 & 4 \\ & & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 29 & -1 \\ -1 & 9 \end{pmatrix}$

$\therefore \underline{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim N_2 \left[\begin{pmatrix} 8 \\ 10 \end{pmatrix}, \begin{pmatrix} 29 & -1 \\ -1 & 9 \end{pmatrix} \right]$

(c) Let $\underline{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + 2 \\ 2x_1 \quad + 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\underline{Y} = A\underline{X} + \underline{a}$$

$$E(\underline{Y}) = A\underline{\mu} + \underline{a}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

$$var(\underline{Y}) = var(A\underline{X} + \underline{a}) = A \Sigma A'$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 & 1 & -2 \\ & 13 & 4 \\ & & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 14 \\ 14 & 24 \end{pmatrix}$$

$$\therefore \underline{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim N_2\left[\begin{pmatrix} 6 \\ 9 \end{pmatrix}, \begin{pmatrix} 21 & 14 \\ 14 & 24 \end{pmatrix}\right.$$