Theorem 3 If the random vector:

$\underline{X}' = (x_1, \dots, x_q, x_{q+1}, \dots, x_p)$ is distributed according to $N_p(\underline{\mu}, \Sigma)$, then the subset of the random variables $X_1, \dots, X_q$ and the subset of the random variables $X_{q+1}, \dots, X_p$ be independent if and only if the covariance between the first and the second subsets of the random variables be zeros.

proof

Let us partition:

$$\underline{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_q \\ \hdashline x_{q+1} \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} \underline{X}_1 \\ \hdashline \underline{X}_2 \end{pmatrix}, \quad \underline{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_q \\ \hdashline \mu_{q+1} \\ \vdots \\ \mu_p \end{pmatrix} = \begin{pmatrix} \underline{\mu}_1 \\ \hdashline \underline{\mu}_2 \end{pmatrix}$$

And $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

such that

$$E(\underline{X}_1) = \underline{\mu}_1, \quad E(\underline{X}_2) = \underline{\mu}_2$$

$$var(\underline{X}_1) = \Sigma_{11} = E(\underline{X}_1 - \underline{\mu}_1)(\underline{X}_1 - \underline{\mu}_1)'$$

$$var(\underline{X}_2) = \Sigma_{22} = E(\underline{X}_2 - \underline{\mu}_2)(\underline{X}_2 - \underline{\mu}_2)'$$

and $cov(\underline{X}_1, \underline{X}_2) = \Sigma_{12} = E(\underline{X}_1 - \underline{\mu}_1)(\underline{X}_2 - \underline{\mu}_2)'$

15/3/1996

Conditional and marginal dist. of multinormal variates.

لدي متجه $\underline{X}$ و يتوزع توزيع طبيعي متعدد.
ماذا لو تم تجزئته كيف يحدث ذلك.

Let x be a vector with multivariate $MVND(\underline{\mu}, \Sigma)$,

$\underset{p\times 1}{\underline{X}}$ has been partitioned into sub vectors $\underset{q\times 1}{X_1}$ and $\underset{(p-q)\times 1}{X_2}$ as:

$$\underline{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_q \\ \hline x_{q+1} \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} X_1 \\ \hline X_2 \end{bmatrix}$$

where $X_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix}$ and $X_2 = \begin{bmatrix} x_{q+1} \\ \vdots \\ x_p \end{bmatrix}$

and $$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_q \\ \hline \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \hline \mu_2 \end{bmatrix}$$

where $\mu_1 = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_q \end{bmatrix}$ and $\mu_2 = \begin{bmatrix} \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix}$

either Σ it is:

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

where $\underset{q\times q}{\Sigma_{11}}$ is var-cov. matrix to set $\underline{X}_1$ and it is $(q\times q)$

$\underset{(p-q)\times(p-q)}{\Sigma_{22}}$ is v-cov matrix to set $\underline{X}_2$ and has $(p-q)\times(p-q)$

$\underset{q\times(p-q)}{\Sigma_{12}}$ is v-cov matrix between X_1 and X_2 $\boxed{Cov(X_1, X_2) = \Sigma_{12}}$

Then $X_1 \sim MVN_p(\underline{\mu}_1, \Sigma_{11})$

$X_2 \sim MVN_p(\underline{\mu}_2, \Sigma_{22})$

independent r. vectors.

(i) Let $\Sigma_{12} = \Sigma_{21}' = 0$ to prove that $\underline{X}_1$ and $\underline{X}_2$ are independent normally distribution. Then

$$\Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$$

It inverse is

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix}$$

Also, we note $|\Sigma| = |\Sigma_{11}| \cdot |\Sigma_{22}|$

$$\Rightarrow |\Sigma|^{\frac{1}{2}} = |\Sigma_{11}|^{\frac{1}{2}} \cdot |\Sigma_{22}|^{\frac{1}{2}} \quad \text{and}$$

$$(2\pi)^{p/2} = (2\pi)^{\frac{q+p-q}{2}} = (2\pi)^{\frac{p-q}{2}} \cdot (2\pi)^{\frac{q}{2}}$$

Thus the quadratic form in the exponent of $n(\underline{x}|\underline{\mu},\Sigma)$ is

$$Q = (\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})$$

$$= \left[(\underline{x}_1-\underline{\mu}_1)', (\underline{x}_2-\underline{\mu}_2)'\right] \begin{bmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix} \begin{bmatrix} (\underline{x}_1-\underline{\mu}_1) \\ (\underline{x}_2-\underline{\mu}_2) \end{bmatrix}$$

$$= \left[(\underline{x}_1-\underline{\mu}_1)' \Sigma_{11}^{-1}, (\underline{x}_2-\underline{\mu}_2)' \Sigma_{22}^{-1}\right] \begin{bmatrix} \underline{x}_1-\underline{\mu}_1 \\ \underline{x}_2-\underline{\mu}_2 \end{bmatrix}$$

$$= (\underline{x}_1-\underline{\mu}_1)' \Sigma_{11}^{-1} (\underline{x}_1-\underline{\mu}_1) + (\underline{x}_2-\underline{\mu}_2)' \Sigma_{22}^{-1} (\underline{x}_2-\underline{\mu}_2)$$

$$= Q_1 + Q_2$$

The density of $\underline{X}$ can be written

$$f(\underline{x}) = \frac{1}{(2\pi)^{\frac{p}{2}} |\Sigma|^{\frac{1}{2}}} e^{-Q/2}$$

$$= \frac{e^{-Q_1/2}}{(2\pi)^{q/2} |\Sigma_{11}|^{\frac{1}{2}}} \cdot \frac{e^{-Q_2/2}}{(2\pi)^{p-q/2} |\Sigma_{22}|^{\frac{1}{2}}}$$

$$= f(\underline{x}_1) \cdot f(\underline{x}_2)$$

Therefore, the two sets of variates are independent $\Longrightarrow$ $\underline{X}_1$ and $\underline{X}_2$ are indep.

(ii) Let $\underline{X}_1$ and $\underline{X}_2$ are independent to prove $\Sigma_{12} = 0$. Then the necessity follows from the fact that if X_i from one set and X_j from the other, then for any density

$$\sigma_{ij} = E(X_i - \mu_i)(X_j - \mu_j)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - \mu_i)(x_j - \mu_j) \cdot f(x_1, \ldots, x_q) f(x_{q+1}, \ldots, x_p)\, dx_1 \cdots dx_p$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - \mu_i) f(x_1, \ldots, x_q)\, dx_1 \cdots dx_q \cdot \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_j - \mu_j) f(x_{q+1}, \ldots, x_p)\, dx_{q+1} \cdots d x_p$$

$$= E(X_i - \mu_i) \cdot E(X_j - \mu_j) = 0$$

$$\Longrightarrow \Sigma_{12} = [\mathrm{cov}(x_i, x_j)] = 0$$

Theorem: The random vectors X_1 and X_2 are independent iff. $\Sigma_{12} = 0$, where $\underline{X}_1 \sim N_p(\underline{\mu}_1, \Sigma_{11})$ and $\underline{X}_2 \sim N_p(\underline{\mu}_2, \Sigma_{22})$.

الاتجاه الاول
هنا نفرض X_1 و X_2 تكون مستقلة فقط اذا كانت $\Sigma_{12} = 0$

proof. To prove the sufficient condition.

(i) → if $\Sigma_{12} = 0$.

to prove that $\underline{X}_1$ and $\underline{X}_2$ are indep.

take

$$\mu = \begin{bmatrix} \mu_1 \\ \hdashline \underline{\mu}_2 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$, because $\Sigma_{12} = 0$. من الفرضية.

then $|\Sigma| = |\Sigma_{11}\,\Sigma_{22} - 0| = |\Sigma_{11}\,\Sigma_{22}| = |\Sigma_{11}| . |\Sigma_{22}|$

$$\therefore \boxed{|\Sigma| = |\Sigma_{11}| . |\Sigma_{22}|}$$

and $$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix}$$, because it is diagonal matrix
لأن Σ مصفوفة قطرية.

Know the Q. F. in $f(x)$ is:
The Quadratic form.

$$Q = (\underline{x} - \underline{\mu}')\,\Sigma^{-1}\,(\underline{x} - \underline{\mu})$$

ملاحظة مهمة
الاستقلالية تعتمد على التباين المشترك (cov.). فاذا كانت صفراً التباين المشترك يساوي فالمتغيرات مستقلة.
وليس على الارتباط correlation

also The Q.F. in $f(\underset{\sim}{x})$ is:

$$Q = (\underset{\sim}{x} - \underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu})$$

$$= \begin{pmatrix} \underset{\sim}{x}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2 - \underset{\sim}{\mu}_2 \end{pmatrix}' \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2 - \underset{\sim}{\mu}_2 \end{pmatrix}$$

$$= \left[(\underset{\sim}{x}_1 - \underset{\sim}{\mu}_1)' \quad (\underset{\sim}{x}_2 - \underset{\sim}{\mu}_2)' \right] \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2 - \underset{\sim}{\mu}_2 \end{pmatrix}$$

$$= \left[(\underset{\sim}{x}_1 - \underset{\sim}{\mu}_1)' \Sigma_{11}^{-1} + 0 \quad \cancel{0+} (\underset{\sim}{x}_2 - \underset{\sim}{\mu}_2)' \Sigma_{22}^{-1} \right] \begin{bmatrix} \underset{\sim}{x}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{x}_2 - \underset{\sim}{\mu}_2 \end{bmatrix}$$

$$Q = \underbrace{(\underset{\sim}{x}_1 - \underset{\sim}{\mu}_1)' \Sigma_{11}^{-1} (\underset{\sim}{x}_1 - \underset{\sim}{\mu}_1)} + \underbrace{(\underset{\sim}{x}_2 - \underset{\sim}{\mu}_2)' \Sigma_{22}^{-1} (\underset{\sim}{x}_2 - \underset{\sim}{\mu}_2)}$$

$$Q = Q_1 + Q_2$$

This means. The density of $\underset{\sim}{x}$ can be written as:

$$f(\underset{\sim}{x}, \underset{\sim}{\mu}, \Sigma) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2} Q}$$

but $Q = Q_1 + Q_2$ then

$$= \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(Q_1 + Q_2)}$$

$$= \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-\frac{1}{2} Q_1} \cdot e^{-\frac{1}{2} Q_2}$$

$$= \frac{1}{(2\pi)^{p/2}\,|\Sigma|^{1/2}}\, e^{-\frac{1}{2}Q_1} \cdot e^{-\frac{1}{2}Q_2}$$

but we have

$$|\Sigma| = |\Sigma_{11}| \cdot |\Sigma_{22}|$$

Then $|\Sigma|^{1/2} = |\Sigma_{11}|^{1/2} \cdot |\Sigma_{22}|^{1/2}$

يعني نأخذ
الجذر التربيعي
للطرفين

$$\sqrt{|\Sigma|} = \sqrt{|\Sigma_{11}| \cdot |\Sigma_{22}|}$$

ولدينا ابعاد.

$\underline{X}_1 = 1 * q$

$\underline{X}_2 = p-q * p$

وبذلك فان

$$\boxed{(2\pi)^{p/2} = (2\pi)^{q/2}\,(2\pi)^{\frac{p-q}{2}}}$$

لانه

$$= (2\pi)^{\frac{q+p-q}{2}} = (2\pi)^{p/2}$$

وبالعودة: الى الصيغة الاصلية

$$f(\underline{X}, \underline{\mu}, \Sigma) = \frac{1}{(2\pi)^{p/2}\,|\Sigma|^{1/2}}\, e^{-\frac{1}{2}Q_1} \cdot e^{-\frac{1}{2}Q_2}$$

$$= \frac{1}{(2\pi)^{\frac{q}{2}} \cdot (2\pi)^{\frac{p-q}{2}}\, |\Sigma_{11}|^{1/2} \cdot |\Sigma_{22}|^{1/2}}\, e^{-\frac{1}{2}Q_1} \cdot e^{-\frac{1}{2}Q_2}$$

$$= \frac{1}{(2\pi)^{\frac{q}{2}}\, |\Sigma_{11}|^{1/2}}\, e^{-\frac{1}{2}Q_1} \cdot \frac{1}{(2\pi)^{\frac{p-q}{2}}\, |\Sigma_{22}|^{1/2}}\, e^{-\frac{1}{2}Q_2}$$

$$f(\underline{X}, \underline{\mu}, \Sigma) = f(\underline{X}_1, \underline{\mu}_1, \Sigma_{11}) \cdot f(\underline{X}_2, \underline{\mu}_2, \Sigma_{22})$$

or $\quad f(\underline{X}) = f(\underline{X}_1) \cdot f(\underline{X}_2)$

There for the two sets ~~af~~ variates are indep.

$\Rightarrow$ X_1 and X_2 are indep.

where

$$\boxed{N_p(\underline{X}, \underline{\mu}, \Sigma) = N_q(\underline{X}_1, \underline{\mu}_1, \Sigma_{11}) \cdot N_{p-q}(\underline{X}_2, \underline{\mu}_2, \Sigma_{22})}$$

Theorem

الحالة الثانية

let X_1 and X_2 are independent.

to prove $\Sigma_{12} = 0$

اذا كانت لدينا

X_1 , X_1 مستقلة

فالمطلوب برهان

بان مصفوفة التباين والتباين المشترك

تساوي صفر .

يعني $\Sigma_{12} = 0.$

from the fact that: if x_i from one the set and x_j from the other set. then for any density we have.

$$f(\underline{x}) = f(x_1) \cdot f(x_2)$$

because X_1 and X_2 are indep,

and $\Sigma_{12} = \{\sigma_{ij}\}$, $\forall i , j = 1, 2, \ldots, p$ and $i \neq j$

Know $\sigma_{ij} = E(x_i - \mu_i)(x_j - \mu_j)$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - \mu_i)(x_j - \mu_j) f(x_1, \ldots, x_p)\, dx_1 \ldots dx_p$$

$E x = \int x f(x)\, dx$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - \mu_i)(x_j - \mu_j) f(x_1, \ldots, x_q) f(x_{q+1}, \ldots, x_p)\, dx_1 \ldots dx_q\, dx_{q+1} \ldots dx_p$$

$f(x) = f(x_1) f$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_j - \mu_j) \underbrace{\left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - \mu_i) f(x_1, \ldots, x_q)\, dx_1, \ldots, dx_q \right]}_{E(x_i - \mu_i)} f(x_{q+1}, \ldots x_p)\, dx_{q+1} \ldots dx_p$$

$$= E(x_i - \mu_i) \underline{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_j - \mu_j) f(x_{q+1}, \ldots x_p)\, dx_{q+1} \ldots dx_p}$$

$$= E(x_i - \mu_i)\; E(x_j - \mu_j)$$

$$= (0) \quad (0)$$

$\sigma_{ij} = 0$ $\longrightarrow$ then $\boxed{\Sigma_{12} = 0}$.

but $E(x_i - \mu_i)$

and $E(x_j - \mu$

or

$$\sigma_{ij} = E(x_i - \mu_j)(x_j - \mu_j)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - \mu_i)(x_j - \mu_j)\, f(\underline{x})\, d\underline{x}$$

$$\Sigma_{12} = \sigma_{ij} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - \mu_i)(x_j - \mu_j)\, f(\underline{x}_1) \cdot f(\underline{x}_2)\, d\underline{x}_1\, d\underline{x}_2$$

because $f(\underline{x}) = f(\underline{x}_1) f(\underline{x}_2)$

$\underline{x}_1$ & $\underline{x}_2$ indep

then

$$\Sigma_{12} = \sigma_{ij} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_j - \mu_j) \left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_i - \mu_i) f(\underline{x}_1)\, d\underline{x}_1 \right] f(\underline{x}_2)\, d\underline{x}_2$$

$$= \quad [E(x_i - \mu_i) = 0]$$

$$= (0) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_j - \mu_j)\, f(\underline{x}_2)\, d\underline{x}_2$$

$$= (0) \quad (E(x_j - \mu_j) = 0)$$

$$= (0) \quad (0)$$

$$\Sigma_{12} = (0)$$

$\therefore$ if x_1 and x_2 are independent then $\Sigma_{12} = 0$

Remark

if $\Sigma_{12} \neq 0$ Can we prove that $\underset{\sim}{X_1}$ and $\underset{\sim}{X_2}$ indep.

means $f(\underset{\sim}{X}) = f(\underset{\sim}{X_1}) \cdot f(\underset{\sim}{X_2})$

proof

since $\underset{\sim}{X_1}$ and $\underset{\sim}{X_2}$ are independent

we can make the following transformation.

$\underset{\sim}{Y_1} = \underset{\sim}{X_1} + C\,\underset{\sim}{X_2}$

$\underset{\sim}{Y_2} = \underset{\sim}{X_2}$

where c is a singular matrix $\ni$ $\underset{\sim}{Y_1}$ and $\underset{\sim}{Y_2}$ are indep.

To find C which make the $\text{Cov}(\underset{\sim}{Y_1}, \underset{\sim}{Y_2}) = 0$

$\underset{\sim}{Y_1}, \underset{\sim}{Y_2}$ indep.

$\Rightarrow$.

prove

$Cov(Y_1, Y_2) = 0$

but $Cov(Y_1, Y_2) = E\left[Y_1 - EY_1\right]\left[Y_2 - EY_2\right]'$

[illegible]

$\Rightarrow \quad E\left[Y_1 - EY_1\right]\left[Y_2 - EY_2\right] = 0$

but

$$\boxed{\begin{aligned} \underline{Y}_1 &= \underline{X}_1 + c\,\underline{X}_2 \\ E\,\underline{Y}_1 &= E\underline{X}_1 + c\,E\underline{X}_2 \\ E\,\underline{Y}_1 &= \underline{\mu}_1 + c\,\underline{\mu}_2 \end{aligned}}$$

and

$$\boxed{\begin{aligned} \underline{Y}_2 &= \underline{X}_2 \\ E\underline{Y}_2 &= E\underline{X}_2 \\ E\underline{Y}_2 &= \underline{\mu}_2 \end{aligned}}$$

where $\underline{X} \sim MVN(\underline{\mu}, \Sigma)$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \underline{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Then.

$$E(\underline{Y}_1 - E\underline{Y}_1)(\underline{Y}_2 - E\underline{Y}_2)' = 0$$

$$E\left[(\underline{X}_1 + c\underline{X}_2) - (\underline{\mu}_1 + c\,\underline{\mu}_2)\right]\left[\underline{X}_2 - \underline{\mu}_2\right]' = 0$$

$$E\left[(\underline{X}_1 + c\underline{X}_2 - \underline{\mu}_1 - c\,\underline{\mu}_2)\right]\left[\underline{X}_2 - \underline{\mu}_2\right]' = 0$$

$$E\left[(\underline{X}_1 - \underline{\mu}_1) + c\underline{X}_2 - c\underline{\mu}_2\right]\left[\underline{X}_2 - \underline{\mu}_2\right]' = 0$$

$$E\left[(\underline{X}_1 - \underline{\mu}_1) + c(\underline{X}_2 - \underline{\mu}_2)\right]\left[\underline{X}_2 - \underline{\mu}_2\right]' = 0$$

$$E\left[(\underline{X}_1 - \underline{\mu}_1)(\underline{X}_2 - \underline{\mu}_2)' + c(X_2 - \mu_2)(X_2 - \mu_2)'\right] = 0$$

$$E(X_1 - \mu_1)(X_2 - \mu_2)' + c\,E(X_2 - \mu_2)(X_2 - \mu_2)' = 0$$

$$\underbrace{E(X_1-\mu_1)(X_2-\mu_2)'}_{\Sigma_{12}} + c\, E(X_2-\mu_2)(X_2-\mu_2)' = 0$$

$$\Sigma_{12} \quad + c \quad \Sigma_{22} \quad = 0$$

نطرح Σ_{12} من الطرفين

$$\Rightarrow \quad c\,\Sigma_{22} = -\Sigma_{12}$$

نضرب في Σ_{22}^{-1} فينتج

$$c\,\Sigma_{22}\Sigma_{22}^{-1} = -\Sigma_{12}\Sigma_{22}^{-1}$$

$$c\,(I) = -\Sigma_{12}\Sigma_{22}^{-1}$$

$$\boxed{c = -\Sigma_{12}\Sigma_{22}^{-1}}$$

That is the values of c which make $\underline{Y}_1$ and $\underline{Y}_2$ uncorrelated

قيمة $\underline{c}$ التي تجعل $\underline{Y}_1$ ، $\underline{Y}_2$ غير مرتبطة.

وبالتعويض بقيمة $\underline{c}$ من المعادلات بين $\underline{X}$ و $\underline{Y}$ ينتج

$$\boxed{\begin{aligned} \underset{q\times 1}{\underline{Y}_1} &= \underset{q\times 1}{\underline{X}_1} - \underset{q\times(p-q)}{\Sigma_{12}}\,\underset{(p-q)\times(p-q)}{\Sigma_{22}^{-1}}\,\underset{(p-q)\times 1}{\underline{X}_2} \\ \underline{Y}_2 &= \underline{X}_2 \end{aligned}}$$

$$\Rightarrow \quad \underline{Y} = \begin{bmatrix} \underline{Y}_1 \\ \cdots \\ \underline{Y}_2 \end{bmatrix} = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \underline{X}_1 \\ \cdots \\ \underline{X}_2 \end{bmatrix}$$

$$\underline{Y} = K \, \underline{X}$$

where $K = \begin{pmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{pmatrix}$

$$\therefore \underset{\sim}{Y} \sim N_p\left(K\underset{\sim}{\mu},\ K\Sigma K'\right)$$

<u>now</u>

$$K\underset{\sim}{\mu} = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{bmatrix}$$

$$K\underset{\sim}{\mu} = \begin{bmatrix} \underset{\sim}{\mu}_1 - \Sigma_{12}\Sigma_{22}^{-1}\underset{\sim}{\mu}_2 \\ \underset{\sim}{\mu}_2 \end{bmatrix}$$

and $$K\Sigma K' = \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\Sigma_{12}\Sigma_{22}^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & \overset{=0}{\begin{gathered} \Sigma_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22} \\ \Sigma_{12} - \Sigma_{12}(I) \\ \Sigma_{12} - \Sigma_{12} = 0 \\ 0 \end{gathered}} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\Sigma_{12}\Sigma_{22}^{-1} & I \end{bmatrix}$$

$$K\Sigma K' = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ \begin{gathered} \Sigma_{21} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \\ \Sigma_{21} - \Sigma_{12}(I) \\ \Sigma_{21} - \Sigma_{12} \\ = 0 \\ 0 \end{gathered} & \Sigma_{22} \end{bmatrix}$$

بسبب ان
$\Sigma_{12} = \Sigma_{21}$

$$\therefore K\Sigma K' = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$

Then

$$\underline{Y} \sim N_p(K\underline{\mu},\ K \Sigma K')$$

$$\underline{Y} = \begin{pmatrix} \underline{Y}_1 \\ Y_2 \end{pmatrix} \sim N_p\left[\begin{pmatrix} \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}\mu_2 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}\right]$$

الى هنا البرهان كان بواسطة التحويل Transformation

الان نحتاج الى إيجاد توزيع ال $\underline{Y}$.

والذي هو.

$$f(\underline{Y}) = f(\underline{Y}_1)\cdot f(Y_2)$$

Y_1 and Y_2 are indep. from. the theorem

من البرهان كان $\underline{Y}_1$ و $\underline{Y}_2$ مستقلة وهذا يعني

This means.

$$\underline{Y}_1 \sim N_q(\underline{\mu}_1 - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}_2,\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

$$Y_2 \sim N_{p-q}(\underline{\mu}_2,\ \Sigma_{22})$$

where $\boxed{\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}$

بعد تحديد التوزيع ال $\underline{Y}$ و ال $\underline{Y}_1$ ، $\underline{Y}_1$

نعود ال العلاقة السابقة.

ونقوم بتحويل ال X . لايجاد .

التوزيع ال X. وكيف

سيكون

$Y_1 = X_1 + X_2$

$Y_2 = X_2$

نجد العلاقة بدلالة $\underline{X}$ و $\underline{Y}$

$$\underline{X}_1 = \underline{Y}_1 + \Sigma_{12}\Sigma_{22}^{-1}\underline{Y}_2$$

$$\underline{X}_2 = \underline{Y}_2$$

$$\underline{X} = \begin{pmatrix} I & \Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{pmatrix}\underline{Y}$$

الصفحة . ←

$$f(\underline{y}) = f(\underline{y}_1)\cdot f(\underline{y}_2)$$

$$f(x_1, x_2) =$$

$$f(x_1,x_2) = \frac{1}{(2\pi)^{q/2}\,|\Sigma_{11\cdot2}|^{1/2}} \exp\left[-\frac{1}{2}\left\{(\underline{x}_1-\underline{\mu}_1)-\Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2-\underline{\mu}_2)\right\}' \Sigma_{11\cdot2}\left\{(\underline{x}_1-\underline{\mu}_1)-\Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2-\underline{\mu}_2)\right\}\right]$$

$$\cdot\frac{1}{(2\pi)^{\frac{p-q}{2}}\,|\Sigma_{22}|^{1/2}}\cdot\exp\left[-\frac{1}{2}(x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2)\right]\cdot|J|$$

$$|J| = \left|\frac{\partial \underline{X}}{\partial \underline{Y}}\right| = \begin{vmatrix} I & \Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{vmatrix} = |I| = 1.$$

وهذا يعني

$$\therefore\ f(\underline{x}_1,\underline{x}_2) = f(\underline{x}_1/\underline{x}_2)\, f(\underline{x}_2)$$

because.

$$f(x_1/x_2) = \frac{1}{(2\pi)^{q/2}\,|\Sigma_{11\cdot2}|^{1/2}}\exp\left[-\frac{1}{2}\left[(x_1-\mu_1)-\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)\right]'\Sigma_{11\cdot2}\left[(x_1-\mu_1)-\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)\right]\right]$$

and

$$f(x_2) = \frac{1}{(2\pi)^{\frac{p-q}{2}}\,|\Sigma_{22}|^{1/2}}\cdot\exp\left[-\frac{1}{2}(x_2-\mu_2)'\Sigma_{22}^{-1}(x_2-\mu_2)\right]$$

وبذلك فان

The conditional dist. of $\underline{X}_1/\underline{X}_2$ is:

$$\underline{X}_1/\underline{X}_2 \sim N_q\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}_2-\underline{\mu}_2),\ \Sigma_{11.2}\right)$$

ويمثل التوزيع الشرطي ال X_1 باستبعاد X_2

حيث ان التوقع الشرطي , المتوسط الشرطي

Conditional mean

$$E(X_1/X_2) = \underline{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}_2-\underline{\mu}_2)$$

وان التباين الشرطي

Conditional variance

$$Var(X_1/X_2) = \Sigma_{11.2}$$

$$= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Theorem 4 If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ and if a set of components of $\underline{X}$ is uncorrelated with the other components, the marginal distribution of the set is multivariate normal with means, variances and covariances obtained by taking the corresponding components of $\underline{\mu}$ and Σ respectively.

Thus $\underline{X}$, $\underline{\mu}$ and Σ partitioned as

$$\underline{X} = \begin{pmatrix} \underline{X}_1 \\ \underline{X}_2 \end{pmatrix}, \quad \underline{\mu} = \begin{pmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

where $\underline{X}_1$ and $\underline{\mu}_1$ are $(q*1)$ and Σ_{11} is $(q*q)$

Then $\underline{X}_1$ is multivariate normal:

If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, then $\underline{X}_1 \sim N_q(\underline{\mu}_1, \Sigma_{11})$

9 **Ex.** Let $\underline{X} \sim N_4\left[\begin{pmatrix} 2 \\ 1 \\ 9 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 & 2 & 0 & 0 \\ & 7 & 0 & 0 \\ & & 8 & 1 \\ & & & 6 \end{pmatrix}\right]$

Find

(a) The distribution of X_1

(b) The distribution of $\underline{X}_2 = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$

(c) Are $\underline{X}_4 = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ and $\underline{X}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ independent?

(d) The distribution of $Y_1 = X_1 - X_3$
$Y_2 = X_2 + X_4$

(e) Are Y_1 and Y_2 independent?

(f) The correlation matrix of $\underline{Y}$.

Sol

(a) According to the theorem (4), we have

$$X_1 \sim N_1(2, 9)$$

(b) According to the theorem (4), we have

$$\underline{X}_2 \sim N_2\left[\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}7 & 0\\0 & 6\end{pmatrix}\right]$$

(c) we have $\sigma_{13} = \sigma_{14} = \sigma_{23} = \sigma_{24} = 0$

$\therefore cov(\underline{X}_1, \underline{X}_4) = 0$

$\longrightarrow$ $\underline{X}_1$ and $\underline{X}_4$ are independent according to the theorem (3).

(d) $$\underline{Y} = \begin{pmatrix}Y_1\\Y_2\end{pmatrix} = \begin{pmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}X_1\\X_2\\X_3\\X_4\end{pmatrix} = C\underline{X}$$

$$E(\underline{Y}) = CE(\underline{X}) = C\underline{\mu} = \begin{pmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}2\\1\\9\\3\end{pmatrix} = \begin{pmatrix}-7\\4\end{pmatrix}$$

$$var(\underline{Y}) = C\Sigma C' = \begin{pmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}9 & 2 & 0 & 0\\2 & 7 & 0 & 0\\0 & 0 & 8 & 1\\0 & 0 & 1 & 6\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\-1 & 0\\0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}17 & 1\\1 & 13\end{pmatrix}$$

$$\Longrightarrow \underline{Y} \sim N_2\left[\begin{pmatrix}-7\\4\end{pmatrix}, \begin{pmatrix}17 & 1\\1 & 13\end{pmatrix}\right]$$

(e) Y_1 and Y_2 are not independent because $\sigma_{12} = 1$

(f) $R = D^{-\frac{1}{2}} \Sigma D^{-\frac{1}{2}}$

$$= \begin{pmatrix} \frac{1}{\sqrt{17}} & 0 \\ 0 & \frac{1}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 17 & 1 \\ 1 & 13 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{17}} & 0 \\ 0 & \frac{1}{\sqrt{13}} \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0.067 \\ 0.067 & 1 \end{pmatrix}$$

example: let $(X_1, X_2) \sim N_2(0, 0, \sigma_1^2, \sigma_2^2, \rho)$

and let $U = \frac{X_1}{\sigma_1} + \frac{X_2}{\sigma_2}$

$V = \frac{X_1}{\sigma_1} - \frac{X_2}{\sigma_2}$

Find: (i) EU, EV

(ii) $V(U)$, $V(V)$

(iii) Show that U and V are independent.

(iv) Find the joint dist. of (U, V)

(i) $E(U) = E\left(\frac{X_1}{\sigma_1} + \frac{X_2}{\sigma_2}\right) = \frac{EX_1}{\sigma_1} + \frac{EX_2}{\sigma_2} = (0) + (0) = \boxed{0}$

$E(V) = E\left(\frac{X_1}{\sigma_1} - \frac{X_2}{\sigma_2}\right) = \frac{EX_1}{\sigma_1} - \frac{EX_2}{\sigma_2} = 0 - 0 = \boxed{0}$

(ii) $V(U) = EU^2 - (EU)^2 = EU^2 - (0) = \boxed{EU^2}$

$= E\left(\frac{X_1}{\sigma_1} + \frac{X_2}{\sigma_2}\right)^2 = E\left(\frac{X_1^2}{\sigma_1^2} + \frac{X_2^2}{\sigma_2^2} + \frac{2X_1X_2}{\sigma_1\sigma_2}\right)$

$= \frac{EX_1^2}{\sigma_1^2} + \frac{EX_2^2}{\sigma_2^2} + 2\frac{EX_1X_2}{\sigma_1\sigma_2}$

but we have $V(X_1) = EX_1^2 - (EX_1)^2$

$V(X_1) = EX_1^2 - 0$

$\boxed{V(X_1) = EX_1^2}$

and $V(X_2) = EX_2^2 - (EX_2)^2$

$V(X_2) = EX_2^2 - 0$

and also

$Cov(X_1, X_2) = EX_1X_2 - EX_1EX_2$

$= EX_1X_2 - (0)(0)$

$\boxed{\sigma_{12} = EX_1X_2}$

$\therefore V(U) = \frac{EX_1^2}{\sigma_1^2} + \frac{EX_2^2}{\sigma_2^2} + \frac{2EX_1X_2}{\sigma_1\sigma_2}$

$$v(u) = \frac{EX_1^2}{\sigma_1^2} + \frac{EX_2^2}{\sigma_2^2} + \frac{2\ EX_1X_2}{\sigma_1\sigma_2}$$

$$= \frac{\sigma_1^2}{\sigma_1^2} + \frac{\sigma_2^2}{\sigma_2^2} + \frac{2\ \sigma_{12}}{\sigma_1\sigma_2}$$

$$= 1 + 1 + 2\frac{\sigma_{12}}{\sigma_1\sigma_2}$$

but $\rho = \frac{Cov(X_1, X_2)}{\sqrt{v(X_1)\ v(X_2)}} = \frac{\sigma_{12}}{\sigma_1\sigma_2}$

Then

$$v(u) = 2 + 2\rho$$

$$\boxed{v(u) = 2(1+\rho)}$$

$$v(v) = E\left(\frac{X_1}{\sigma_1} - \frac{X_2}{\sigma_2}\right)^2$$

$$= E\left(\frac{X_1^2}{\sigma_1^2} + \frac{X_2^2}{\sigma_2^2} - 2\frac{X_1X_2}{\sigma_1\sigma_2}\right)$$

$$= \frac{EX_1^2}{\sigma_1^2} + \frac{EX_2^2}{\sigma_2^2} - \frac{2\ EX_1X_2}{\sigma_1\sigma_2}$$

$$= \frac{\sigma_1^2}{\sigma_1^2} + \frac{\sigma_2^2}{\sigma_2^2} - 2\frac{\sigma_{12}}{\sigma_1\sigma_2}$$

$$= 1 + 1 - 2\rho$$

$$= 2 - 2\rho$$

$$\boxed{v(v) = 2(1-\rho)}$$

(iii) U and V indep.
iff. $Cov(U,V)=0$.

$Cov(U,V) = E\,UV - EU.\,EV$

but $EU=0$, $E(V)=0$

then

$$\begin{aligned}
Cov(U,V) &= E(U.V) \\
&= E\left(\frac{X_1}{\sigma_1}+\frac{X_2}{\sigma_2}\right)\left(\frac{X_1}{\sigma_1}-\frac{X_2}{\sigma_2}\right) \\
&= E\left(\frac{X_1^2}{\sigma_1^2}-\cancel{\frac{X_1X_2}{\sigma_1\sigma_2}}+\cancel{\frac{X_2X_1}{\sigma_2\sigma_1}}-\frac{X_2^2}{\sigma_2^2}\right) \\
&= E\left(\frac{X_1^2}{\sigma_1^2}-\frac{X_2^2}{\sigma_2^2}\right) \\
&= \frac{EX_1^2}{\sigma_1^2}-\frac{EX_2^2}{\sigma_2^2} \\
&= 1-1
\end{aligned}$$

$Cov(U,V) = 0$

Then U and V are independent.

(iv) joint dist. of u and v.

$g(u,v) = g(u) \cdot g(v)$. because u & v are indep.

~~Then~~ but. $g(u) = \frac{1}{\sqrt{2\pi}\,(2+2\rho)^{1/2}} \exp\left[-\frac{1}{2}\frac{u^2}{(2+2\rho)}\right]$

$g(v) = \frac{1}{\sqrt{2\pi}\,(2-2\rho)^{1/2}} \exp\left[-\frac{1}{2}\frac{v^2}{(2-2\rho)}\right]$

$$g(u,v) = g(u) \cdot g(v)$$

$$= \frac{1}{\sqrt{2\pi}\,(2+2\rho)^{1/2}} e^{-\frac{u^2}{2(2+2\rho)}} \cdot \frac{1}{\sqrt{2\pi}\,(2-2\rho)^{1/2}} e^{-\frac{v^2}{2(2-2\rho)}}$$

$$= \frac{1}{2\pi\,(2+2\rho)^{1/2}(2-2\rho)^{1/2}} e^{-\frac{1}{2}\left(\frac{u^2}{(2+2\rho)} - \frac{v^2}{(2-2\rho)}\right)}$$

$$= \frac{1}{2\pi\sqrt{(2+2\rho)(2-2\rho)}} \exp\left\{-\frac{1}{2}\left[\begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} \frac{1}{2+2\rho} & 0 \\ 0 & \frac{1}{2-2\rho} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\right]\right\}$$

$$g(u,v) = \frac{1}{2\pi\sqrt{4-\cancel{4\rho}+\cancel{4\rho}+4\rho^2}} \exp\left[-\frac{1}{2}\begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} \frac{1}{2+2\rho} & 0 \\ 0 & \frac{1}{2-2\rho} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\right]$$

$$g(u,v) = \frac{1}{2\pi\,(4-4\rho^2)^{1/2}} \exp\left\{-\frac{1}{2}\begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} \frac{1}{2+2\rho} & 0 \\ 0 & \frac{1}{2-2\rho} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\right\}$$

$$g(u,v) = \frac{1}{2\pi\; 2\,(1-\rho^2)^{1/2}} \exp\left[-\frac{1}{2}\begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} (2+2\rho) & 0 \\ 0 & 2-2\rho \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\right]$$

$$\therefore \begin{pmatrix} u \\ v \end{pmatrix} \sim N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2+2\rho & 0 \\ 0 & 2-2\rho \end{pmatrix}\right]$$

التوزيع الشرطي لمتعدد المتغيرات الطبيعي

Conditional dist. for MVND.

عرفنا بان الاستقلالية بين المتغيرات.
تحدث عندما تكون التباين المشترك فيما
بينهما يساوي صفر بمعنى اي X_1 و X_2
مستقلات اذا كانت $\Sigma_{12} = 0$

فاذا كانت $\Sigma_{12} \neq 0$ اي التباين المشترك.
لايساوي صفر. فهذا يعني بان X_1 سيعتمد على X_2
وكذلك فان X_2 سيعتمد على X_1.
اي بمعنى آخر يوجد توزيع جديد شرطي وشرطه
انه سيعتمد على المتغير الاخر.
وهو مرتبط معه.

تقاطع

فمثلاً في الاحتمالات. $p(A/B) = \dfrac{p(AB)}{p(B)}$

وتعني احتمالية الحدث A عندما تكون B معلومة.

اما في المتغيرات مثل X_1 و X_2

$$f(X_1/X_2) = \frac{f(X_1, X_2)}{f(X_2)} = \frac{\text{joint}}{\text{marginal}}$$ التوزيع الشرطي.

فهو يعتمد بشكل وثيق على ان X_1 و X_2 غير مستقلة.
اي التباين المشترك فيما بينهما لايساوي صفر.

اما في المتعدد. سنعتمد على مصفوفة التباين والتباين المشترك

فاذا $\Sigma_{12} = 0$ فالمتغيرات مستقلة.

واذا $\Sigma_{12} \neq 0$ فالمتغيرات غير مستقلة.

Theorem 5:

let X be $N_p(\underline{\mu}, \Sigma)$

let us $\underline{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_q \\ \hdashline X_{q+1} \\ \vdots \\ X_p \end{pmatrix} = \begin{pmatrix} \underline{X}_1 \\ \hdashline \underline{X}_2 \end{pmatrix} \begin{matrix} q\times 1 \\ \\ (p-q)\times 1 \end{matrix}$

and $\underline{\mu} = \begin{pmatrix} \underline{\mu}_1 \\ \hdashline \underline{\mu}_2 \end{pmatrix}$

$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

Then the conditional dist. of $\underline{X}_1$ given $\underline{X}_2$ is normal dist. with.

$E(\underline{X}_1/\underline{X}_2) = \underline{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}_2 - \underline{\mu}_2)$

$Var(\underline{X}_1/\underline{X}_2) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

Remark

(1) it should be noted that $E(\underline{X}_1/\underline{X}_2)$ is called Conditional mean and $var(\underline{X}_1/\underline{X}_2)$ is called Conditional variance

(2) we noted that the mean $E(\underline{X}_1/\underline{X}_2)$ of the density $f(\underline{x}_1/\underline{x}_2)$ is simply alinear function of $\underline{X}_2$, and the variance $(\underline{X}_1/\underline{X}_2)$ does not depend on $\underline{X}_2$ at all.

(3) The density $f(\underline{X}_2/\underline{X}_1)$ is a $(p-q)$ - variate normal dist. with

$$E(\underline{X}_2/\underline{X}_1) = \underline{\mu}_2 + \Sigma_{21}\Sigma_{11}^{-1}(\underline{X}_1 - \underline{\mu}_1)$$

$$Var(\underline{X}_2/\underline{X}_1) = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$$

where $\Sigma_{12} = \Sigma_{21}$

Then also

$$\underline{X}_2/\underline{X}_1 \sim N_{p-q}\left(\underline{\mu}_2 + \Sigma_{21}\Sigma_{11}^{-1}(\underline{X}_1 - \underline{\mu}_1),\ \Sigma_{22.1}\right)$$

where $\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$

(4) The vector $E(\underline{X}_1/\underline{X}_2)$ is called the regression fun.

التوقع الشرطي لـ X_1 اذا كان X_2 يسمى بدالة الانحدار.

(5) The matrix $\Sigma_{12}\Sigma_{22}^{-1}$ is the matrix of regression cofficie of $\underline{X}_1$ and $\underline{X}_2$ and denoted by $\boxed{B}$ where

$$\boxed{B = \Sigma_{12}\Sigma_{22}^{-1}}$$

ومصفوفة معاملات الانحدار.

(6) let $\underline{X}_{1.2}$ be arandom vector of residual variates then.

$$\underline{X}_{1.2} = X_1 - E(\underline{X}_1/\underline{X}_2)$$

البواقي = قيمة حقيقية - قيمة تقديرية

$\underline{X}_{1.2}$
عن متغيرات البواقي
ومن خصائصها ← (1) (2) (3)

since (1) $E(\underline{X}_{1.2}) = 0$

(2) $Cov(\underline{X}_2, \underline{X}_{1.2}) = 0$

(3) $Cov(\underline{X}_1, \underline{X}_{1.2}) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ is the v-Covariance matrix of the Conditional dist.

Theorem 5:

if $\underset{\sim}{X} \sim MVD(\underline{\mu}, \Sigma)$. the random vector $\underline{X}$ ~~denoted~~ divided into two subvectors $\underline{X}_1$ and $\underline{X}_2$ such that.

$$\underset{\sim}{X} = \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix}, \quad \mu = \begin{bmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix}$$

Then the Conditional dist. of x_1 given ~~that~~ the elements of $\underline{X}_2 = x_2$ is multivariate with:

$$E(\underline{X}_1 / \underline{X}_2) = \underline{\mu}_1 + \Sigma_{12} \Sigma_{22}^{-1} (\cancel{\underline{X}}_2 - \underline{\mu}_2) \rightarrow \text{mean vector}$$

المتوسط الشرطي.

$$V(\underline{X}_1 / \underline{X}_2) = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \rightarrow \text{v-c matrix}$$

مصفوفة التباين والتباين المشترك الشرطية.

ملاحظة

مايؤخذ من الهامشي للتوزيع الشرطي و المؤدي للمارجينل

هو ضمني للعلاقات الخطية

المكونات عندما تجزئة فتنتج

proof.

$$g(\underline{x}_1 / \underline{x}_2) = \frac{f(\underline{x}_1, \underline{x}_2)}{h(\underline{x}_2)}$$

$$h(\underline{x}_2) = \frac{1}{(2\pi)^{q/2} \, |\Sigma_{22}|^{1/2}} \exp\left\{ -\frac{1}{2} (\underline{x}_2 - \underline{\mu}_2)' \, \Sigma_{22}^{-1} (\underline{x}_2 - \underline{\mu}_2) \right\}$$

set

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

where $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix}$

then $\Sigma^{-1} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix}^{-1}$

and $|\Sigma| = |\Sigma_{22}| \cdot |\Sigma_{11.2}|$

توضيح.

للمعلومات، $|\Sigma| = |\Sigma_{11}| \cdot |\Sigma_{22\cdot}$

$|A| = |A_{11}| \cdot |A_{22} - A_{21} A_{11}^{-1} A_{12}|$

$|A| = |A_{22}| \cdot |A_{11} - A_{12} A_{22}^{-1} A_{21}|$

$\therefore |\Sigma| = |\Sigma_{22}| \, |\Sigma_{11.2}|$

ان $\Sigma_{11.2}$ تمثل مصفوفة التباين والتباين المشترك الشرطية
باعتبار ان احدى العناصر تكون متغير، والاخرى ثابته

$$\underline{x} - \underline{\mu} = \begin{pmatrix} \underline{x}_1 - \underline{\mu}_1 \\ \underline{x}_2 - \underline{\mu}_2 \end{pmatrix} = \begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \end{pmatrix} = \underline{y}$$

$$f(\underline{x}) = \frac{1}{(2\pi)^{\frac{p+q}{2}} \, |\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2} (\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu}) \right\}$$

$$= \frac{1}{(2\pi)^{\frac{p+q}{2}} \, |\Sigma|^{1/2}} \exp\left[-\frac{1}{2} \, y' \Sigma^{-1} y \right\}$$

$$\Rightarrow y'\Sigma^{-1} y = [\underline{y}_1' \;\; \underline{y}_2'] \begin{pmatrix} \Sigma_{11.2}^{-1} & -\Sigma_{11.2}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \\ -\Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11.2}^{-1} & \Sigma_{22}^{-1} + \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11.2}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \end{pmatrix} ($$

$$= \Big(\underline{y}_1' \Sigma_{11.2}^{-1} - \underline{y}_2' \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11.2}^{-1} \quad - \underline{y}_1' \Sigma_{11.2}^{-1} \Sigma_{12} \Sigma_{22}^{-1} + \underline{y}_2' \Sigma_{22}^{-1} + \underline{y}_2' \Sigma_{22}^{-1} \Sigma_{21} \ldots \Sigma_{12} \ldots$$

$$= \Big(\underline{y}_1' \Sigma_{11.2}^{-1} \underline{y}_1 - \underline{y}_2' \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11.2}^{-1} \underline{y}_1 - \underline{y}_1' \Sigma_{11.2}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \underline{y}_2 + \underline{y}_2' \Sigma_{22}^{-1} \underline{y}_2 + \underline{y}_2' \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11.2}^{-1} \Sigma_{12} \ldots$$

$$= \Big(\underline{y}_2' \Sigma_{22}^{-1} \underline{y}_2 + \underline{y}_1' \Sigma_{11.2}^{-1} (\underline{y}_1 - \Sigma_{12} \Sigma_{22}^{-1} \underline{y}_2) - \underline{y}_2' \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11.2}^{-1} (\underline{y}_1 - \Sigma_{12} \Sigma_{22}^{-1} \underline{y}_2) \Big)$$

$$= \Big[\underline{y}_2' \Sigma_{22}^{-1} \underline{y}_2 + (\underline{y}_1 - \Sigma_{12} \Sigma_{22}^{-1} \underline{y}_2) (\underline{y}_1' \Sigma_{11.2}^{-1} - \underline{y}_2' \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11.2}^{-1}) \Big]$$

$$= \underline{y}_2' \Sigma_{22}^{-1} y_2 + (y_1 - \Sigma_{12} \Sigma_{22}^{-1} y_2) \; \Sigma_{11.2}^{-1} (y_1' - \cancel{y_2} \, y_2' \Sigma_{22}^{-1} \Sigma_{21})$$

$$y' \Sigma^{-1} y. = \underline{y}_2' \Sigma_{22}^{-1} \underline{y}_2 + (\underline{y}_1 - \Sigma_{12} \Sigma_{22}^{-1} \underline{y}_2)' \, \Sigma_{11.2}^{-1} (\underline{y}_1 - \Sigma_{12} \Sigma_{22}^{-1} \underline{y}_2)$$

Let $\Theta = \Sigma_{12} \Sigma_{22}^{-1} \underline{y}_2$

Then.

Then

$$\underline{y}'\Sigma^{-1}\underline{y} = y_2'\Sigma_{22}^{-1}y_2 + (\underline{y}_1-\theta)'\Sigma_{11.2}^{-1}(\underline{y}_1-\theta)$$

$$\therefore \quad g(\underline{x}_1/\underline{x}_2) = \frac{\dfrac{1}{(2\pi)^{\frac{p+q}{2}}\cancel{|\Sigma|^{1/2}}\;|\Sigma_{11.2}|}\exp\left\{-\frac{1}{2}\left(\cancel{y_2'\Sigma_{22}^{-1}y_2} + (\underline{y}_1-\theta)'\Sigma_{11.2}^{-1}(\underline{y}_1-\theta)\right)\right\}}{\dfrac{1}{(2\pi)^{\frac{q}{2}}\cancel{|\Sigma_{22}|^{1/2}}}\exp\left\{-\frac{1}{2}\cancel{y_2'\Sigma_{22}^{-1}y_2}\right\}}$$

because $g(\underline{x}_1/\underline{x}_2) = \dfrac{f(\underline{x}_1,\underline{x}_2)}{h(\underline{x}_2)}$

and we have $|\Sigma| = |\Sigma_{22}|\,.\,|\Sigma_{11.2}|$

Then.

$$\frac{|\Sigma|^{1/2}}{|\Sigma_{22}|^{1/2}} = \frac{\cancel{|\Sigma_{22}|^{1/2}}\,|\Sigma_{11.2}|^{1/2}}{\cancel{|\Sigma_{22}|^{1/2}}}$$

Then:

$$g(\underline{x}_1/\underline{x}_2) = \frac{1}{(2\pi)^{\frac{p+\cancel{q}-\cancel{q}}{2}}|\Sigma_{11.2}|^{1/2}}\exp\left\{-\frac{1}{2}(\underline{y}_1-\theta)'\Sigma_{11.2}^{-1}(\underline{y}_1-\theta)\right\}$$

$$g(\underline{x}_1/\underline{x}_2) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma_{11.2}|^{1/2}}\exp\left\{-\frac{1}{2}(\underline{y}_1-\theta)'\Sigma_{11.2}^{-1}(\underline{y}_1-\theta)\right\}$$

Know

$$\underline{y}_1-\theta = \underbrace{(x_1-\mu_1)}_{\underline{y}_1} \underbrace{-}_{-} \underbrace{\Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2-\underline{\mu}_2)}_{\theta}$$

$$\underline{y}_1-\theta = \underline{x}_1 - \underbrace{(\underline{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2-\underline{\mu}_2)}_{\text{عيل التوقع الشرطي}}$$

mean conditional vector

التوقع الشرطي

$$\therefore \boxed{\begin{aligned} &\mu_{\underline{x}_1/\underline{x}_2} = E(\underline{x}_1/\underline{x}_2) = \underline{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\underline{x}_2-\underline{\mu}_2) \\ &\text{J-Covariance matrix} = \Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \end{aligned}}$$

التباين والتباين المشترك الشرطي

Note

The mean of $\underline{X}_1$ given $\underline{X}_2$
is alinear function of $\underline{X}_2$
and the v- covariance matrix
of $\underline{X}_1$ given $\underline{X}_2$ does not depend
on $\underline{X}_2$ at all.

المتوسط الشرطي هو دالة من $\underline{X}_2$
اي انه يتأثر بـ $\underline{X}_2$ بينما مصفوفة
التباين والتباين المشترك الشرطية.
لا تتأثر لانها خالية من $\underline{X}_2$
بينما الوسيط دالة من $\underline{X}_2$ تتأثر بالكيفن

example.

if $X \sim N_3(\underline{\mu}, \Sigma)$, where $\underline{\mu} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{pmatrix}$

Find (a) The p.d.f. of $\underline{y} = X_1 - 3X_2 + 2X_3$

(b) The joint of $\underline{X}_1$ where $X_1 = \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$

(c) The Conditional mean $E(X_1 / X_2, X_3)$

(d). The Simple Correlation r_{13}

من المحاضرة الرابعة
محاضرة ٢٠٢٠
١٨/٣/١٩٩٦
هذا الواجب
محاضرة

if $\underline{x} \sim N_3(\underline{\mu}, \Sigma)$, where $\underline{\mu} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{pmatrix}$

find (a) The p.d.f. of

$$y = x_1 - 3x_2 + 2x_3$$

sol

$$y = x_1 - 3x_2 + 2x_3$$

$$y = (1 \quad -3 \quad 2)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \boxed{\underline{y} = c'\underline{x}}$$

نجد التوقع

$$\Rightarrow c'\mu = (1 \quad -3 \quad 2)\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = 3+3+0 = \boxed{6}$$

نجد التباين

$$\Rightarrow c'\Sigma c = (1 \quad -3 \quad 2)\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = (8 \quad -3 \quad 13)\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \boxed{43}$$

$$\because \quad Y \sim N(c'\mu, c'\Sigma c)$$

The p.d.f. of Y is

$$f(y) = \frac{1}{(2\pi)^{q/2}\,|c'\Sigma c|^{1/2}}\, e^{-\frac{1}{2}(y-c'\mu)'(c'\Sigma c)^{-1}(y-c'\mu)}$$

But we find that $c'\mu = 6$ and $c'\Sigma c = 43$

then

$$Y \sim N(6, 43)$$

and also then

$$f(y) = \frac{1}{(2\pi)^{1/2}\sqrt{43}}\, e^{-\frac{1}{2}\frac{(y-6)^2}{(43)}}$$

, it is the p.d.f. of y.

(b) The joint of X_1 where $\underline{X}_1 = \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$

$$(X_1, X_3) \sim N_2\left(\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \right)$$

This is the joint of X_1

where $\underline{X}_1 = \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$.

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$ صعب ا

ونحن نريد فقط

ال X_1 و X_3 وهذا يعني

$$\mu_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\Sigma = \begin{array}{c} \begin{array}{ccc} 11 & & 13 \end{array} \\ \begin{pmatrix} 2 & \cancel{} & 3 \\ \cancel{0} & \cancel{} & \cancel{0} \\ 3 & \cancel{} & 5 \end{pmatrix} \\ \begin{array}{ccc} 31 & & 33 \end{array} \end{array}$$ اما

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{13} \\ \Sigma_{31} & \Sigma_{33} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

(c) The Conditional mean $E(X_1 \mid X_2 X_3)$.

we have

$$E(\underline{X}_1 / \underline{X}_2) = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2).$$

Then we suppose that.

$\underline{X}_1 = X_1$ Then $\underline{\mu}_1 = \underline{\mu}_1 = 3$

$\underline{X}_2 = \begin{pmatrix} X_2 \\ X_3 \end{pmatrix}$ $\mu_2 = \begin{pmatrix} \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

and $$\Sigma = \left(\begin{array}{c|cc} 2 & 0 & 3 \\ \hline 0 & 1 & 0 \\ 3 & 0 & 5 \end{array}\right)$$

This is $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$

Then $\Sigma_{11} = 2$

$\Sigma_{12} = (0 \quad 3)$

$\Sigma_{21} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$\Sigma_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

$\Rightarrow \Sigma_{22}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$

$$\therefore E(X_1/X_2) = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2)$$

$$= 3 + (0 \quad 3)\begin{pmatrix} 1 & 0 \\ 0 & 1/5 \end{pmatrix}\begin{pmatrix} X_2 + 1 \\ X_3 - 0 \end{pmatrix}$$

$$= 3 + \left(0 \quad \tfrac{3}{5}\right)\begin{pmatrix} X_2 + 1 \\ X_3 \end{pmatrix}$$

$$= 3 + 0 + \tfrac{3}{5}X_3$$

$$E(X_1/X_2) = 3 + \tfrac{3}{5}X_3$$

$$\therefore \boxed{E(X_1 / X_2 X_3) = 3 + \tfrac{3}{5}X_3}$$

. ر.ه.ع

كذلك يمكن حساب $\Sigma_{11.2}$ كالآتي

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21}$$

$$= 2 - \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/5 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$= 2 - \begin{pmatrix} 0 & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$= 2 - \frac{9}{5}.$$

$$= \frac{10-9}{5}$$

$$\boxed{\Sigma_{11.2} = \frac{1}{5}}.$$

وبذلك فإن التوزيع ال X_1/X_2X_3 يكون كالآتي.

$$X_1/X_2X_3 \sim N_1\left(\frac{3}{5}X_3+3\,,\ \frac{1}{5}\right)$$

(d) The simple Correlation r_{13}.

$$r_{13} = \frac{\sigma_{13}}{\sigma_1 \cdot \sigma_3} = \frac{\sigma_{13}}{\sqrt{\sigma_{11}}\,\sqrt{\sigma_{33}}}$$

as we know that.

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{pmatrix}$$

Then $\sigma_{13} = 3$ and $\sigma_{11} = 2$, $\sigma_{33} = 5$

Therefore

$$r_{13} = \frac{\sigma_{13}}{\sqrt{\sigma_{11}}\,\sqrt{\sigma_{33}}} = \frac{3}{\sqrt{2}\,\sqrt{5}} = \frac{3}{\sqrt{10}}$$

Ex)

let $\underset{\sim}{X} \sim N_3(\underset{\sim}{\mu}, \Sigma)$

with $\mu = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 7 & 3 & -3 \\ & 6 & 0 \\ & & 5 \end{pmatrix}$

Find

(a) The conditional dist. of $\underset{\sim}{X}_1 = x_1$ given $\underset{\sim}{X}_2' = (x_2 \;\; x_3)$

(b) The Conditional dist. of $\underset{\sim}{X}_1' = (x_1 \;\; x_3)$ given $\underset{\sim}{X}_2 = x_2$

ⓐ Sol

$$\Sigma = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{c|cc} 7 & 3 & -3 \\ \hline 3 & 6 & 0 \\ -3 & 0 & 5 \end{array}\right) \end{array} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \underline{\mu}_2 \end{pmatrix} = \begin{pmatrix} 2 \\ \hline -1 \\ 3 \end{pmatrix}$$

$$E(\underline{X}_1 / \underline{x}_2) = \underline{\mu}_1 + \Sigma_{12} \Sigma_{22}^{-1} (\underline{x}_2 - \underline{\mu}_2)$$

$$= 2 + (3 \ \ -3) \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \cancel{\begin{pmatrix} 3 \\ -3 \end{pmatrix}} \begin{pmatrix} x_2 + 1 \\ x_3 - 3 \end{pmatrix}$$

$$= \frac{1}{2} x_2 - \frac{3}{5} x_3 + \frac{43}{10}$$

$$\operatorname{var}(\underline{X}_1 / \underline{x}_2) = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

$$= 7 - (3 \ \ -3) \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

$$= \frac{37}{10}$$

$$\therefore (\underline{X}_1 / \underline{x}_2) \sim N_1 \left[\left(\frac{1}{2} x_2 - \frac{3}{5} x_3 + \frac{43}{10} \right), \frac{37}{10} \right]$$

ⓑ

$$\Sigma = \begin{array}{c} \begin{array}{ccc} x_1 & x_3 & x_2 \end{array} \\ \left(\begin{array}{cc|c} 7 & -3 & 3 \\ -3 & 5 & 0 \\ \hline 3 & 0 & 6 \end{array}\right) \end{array} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

$$E(\underline{X}_1 / \underline{x}_2) = \underline{\mu}_1 + \Sigma_{12} \Sigma_{22}^{-1} (\underline{x}_2 - \underline{\mu}_2)$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \frac{1}{6} (x_2 + 1)$$

$$= \begin{pmatrix} \frac{3}{2} + \frac{x_2}{2} \\ 3 \end{pmatrix}$$

$$var(\underline{X}_1/\underline{x}_2) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$= \begin{pmatrix} 7 & -3 \\ -3 & 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -3 \\ -3 & 5 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5.5 & -3 \\ -3 & 5 \end{pmatrix}$$

$$\therefore (\underline{X}_1/\underline{x}_2) \sim N_2\left[\begin{pmatrix} \frac{3}{2} + \frac{x_2}{2} \\ 3 \end{pmatrix}, \begin{pmatrix} 5.5 & -3 \\ -3 & 5 \end{pmatrix}\right]$$

H.W Let $\underline{X} \sim N_3(\underline{\mu}, \Sigma)$ with

$$\underline{\mu} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 9 & 2 & 0 \\ & 7 & 0 \\ & & 8 \end{pmatrix}$$

Find

(a) The conditional distribution of $\underline{X}_1 = x$ given $\underline{X}_2' = (x_1 \;\; x_2)$

(b) The conditional distribution of $\underline{X}_2 = x_2$ given $\underline{X}_1' = (x_1 \;\; x_3)$

<u>example</u>.

Suppose that $X \sim N_3(\underline{\mu}, \Sigma)$

where $\underline{\mu} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 6 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

Find

(1) The marginal dist. of X_1, X_2 and X_3 ص46

(2) The joint dist. of (X_1, X_2), (X_2, X_3) ص46

(3) The conditional dist. of ص46 ص47

(a) $(X_1, X_2)/X_3$ ص47

(b) $(X_1, X_3)/X_2$ ص48 ص49

(c) $(X_2, X_3)/X_1$ ص50

(4) The dist. of $Y = 2X_1 - X_2 + X_3$ ص51

(5) The dist. of $(X_1 / X_2 X_3)$ ص51

Theorem 6. The characteristic function of $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ is:

$$\Phi_{\underline{X}}(\underline{t}) = E\, e^{i\underline{t}'\underline{X}} = e^{i\underline{t}'\underline{\mu} - \frac{1}{2}\underline{t}'\Sigma\underline{t}}$$

For every real vector $\underline{t}$, where $i = \sqrt{-1}$, $\underline{t}' = (t_1, \ldots, t_p)$

proof: Since Σ is positive definite matrix $\exists$ nonsingular $C \ni \Sigma = CC'$

Let $C\underline{Y} = \underline{X} - \underline{\mu}$ or $\underline{Y} = C^{-1}(\underline{X} - \underline{\mu})$

Thus $E(\underline{Y}) = C^{-1}E(\underline{X} - \underline{\mu}) = C^{-1}(\mu - \mu)$
$= 0$

$var(\underline{Y}) = C^{-1}\, var(\underline{X} - \underline{\mu})\, (C^{-1})'$
$= C^{-1}\,\Sigma\, C^{-1'}$

Since $\Sigma = CC'$, then

$var(\underline{Y}) = C^{-1}C\,C'\,C^{-1'} = I$

$\therefore \underline{Y} \sim N_p(\underline{0}, I)$ and

$y_j \sim N(0, 1)$

$\Phi_{y_j}(t_j) = e^{-\frac{1}{2}t_j^2}$

Then $\Phi_{\underline{Y}}(\underline{t}) = E\, e^{i\underline{t}'\underline{Y}} = E\, e^{i(t_1, \ldots, t_p)\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}}$

$$= E\, e^{it_1y_1 + it_2y_2 + \cdots + it_py_p} = E\, e^{it_1y_1} \cdots E\, e^{it_py_p}$$

$$\phi_{\underline{Y}}(\underline{t}) = \prod_{j=1}^{p} E\, e^{it_jy_j} = \prod_{j=1}^{p} e^{-\frac{1}{2}t_j^2}$$

$$= e^{-\frac{1}{2}\underline{t}'\underline{t}}$$

Thus

$$\phi_{\underline{X}}(\underline{t}) = E\, e^{i\underline{t}'(C\underline{Y} + \underline{\mu})}$$

$$= E\, e^{i\underline{t}'C\underline{Y} + i\underline{t}'\underline{\mu}}$$

$$= e^{i\underline{t}'\underline{\mu}}\, E\, e^{i\underline{t}'C\underline{Y}}$$

$$= e^{i\underline{t}'\underline{\mu}}\, \phi_{\underline{Y}}(C'\underline{t})$$

$$= e^{i\underline{t}'\underline{\mu}}\, e^{-\frac{1}{2}(C'\underline{t})'(C'\underline{t})}$$

$$= e^{i\underline{t}'\underline{\mu} - \frac{1}{2}\underline{t}'CC'\underline{t}}$$

$$= e^{i\underline{t}'\underline{\mu} - \frac{1}{2}\underline{t}'\Sigma\underline{t}}$$

Theorem 7. If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$. The moment generating function of $\underline{X}$ is

$$M_{\underline{X}}(\underline{t}) = E\, e^{\underline{t}'\underline{X}} = e^{\underline{t}'\underline{\mu} + \frac{1}{2}\underline{t}'\Sigma\underline{t}}$$

Remark: The mean vector and the covariance matrix of the random vector $\underline{X}$ can be obtained from the moment generating function. The mean vector is

$$\overset{*}{\mu}_{\underline{X}}(\underline{t}) = \ln \mu_{\underline{X}}(\underline{t}) = \underline{t}'\underline{\mu} + \frac{1}{2}\underline{t}'\Sigma \underline{t}$$

$$\overset{*}{\mu}{}'_{\underline{X}}(\underline{t}) = \frac{\partial \overset{*}{\mu}_{\underline{X}}(\underline{t})}{\partial \underline{t}} = \underline{\mu} + \Sigma \underline{t}$$

$$\Rightarrow \overset{*}{\mu}{}'_{\underline{X}}(\underline{0}) = E\underline{X} = \underline{\mu}$$

$$\overset{*}{\mu}{}''_{\underline{X}}(\underline{t}) = \frac{\partial^2 \overset{*}{\mu}_{\underline{X}}(\underline{t})}{\partial \underline{t}\,\underline{t}'} = \Sigma + \underline{0}$$

$$\Rightarrow \overset{*}{\mu}{}''_{X}(\underline{0}) = E(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})' = \Sigma$$

Ex: If $\underline{X} \sim N_2(\underline{0}, I)$. Find the dist^n of $\underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}X_1 \\ X_2 + 3 \end{pmatrix}$ by using the characteristic function

Sol: $\because \underline{X} \sim N_2(\underline{0}, I)$

Sol $\therefore \underline{X} \sim N_2(\underline{0}, I)$

$$\therefore \Phi_{\underline{X}}(\underline{t}) = E\, e^{i \underline{t}' \underline{X}} = e^{i \underline{t}' \underline{\mu} - \frac{1}{2} \underline{t}' \Sigma \underline{t}}$$

$$= e^{-\frac{1}{2} \underline{t}' \underline{t}}$$

Since $\underline{Y} = \begin{pmatrix} \frac{1}{2} x_1 \\ x_2 + 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$\rightarrow \underline{Y} = C \underline{X} + \underline{d}$, where $\underline{d} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$$\Phi_{\underline{Y}}(\underline{t}) = E\, e^{i \underline{t}' \underline{Y}} = E\, e^{i \underline{t}' (C\underline{X} + \underline{d})}$$

$$= E\, e^{i \underline{t}' C \underline{X} + i \underline{t}' \underline{d}} = e^{i \underline{t}' \underline{d}}\, E\, e^{i \underline{t}' C \underline{X}}$$

$$= e^{i \underline{t}' \underline{d}} \cdot e^{-\frac{1}{2} (C' \underline{t})' (C' \underline{t})}$$

$$= e^{i \underline{t}' \underline{d} - \frac{1}{2} \underline{t}' C C' \underline{t}}$$

$$= e^{i \underline{t}' \begin{pmatrix} 0 \\ 3 \end{pmatrix} - \frac{1}{2} \underline{t}' \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix} \underline{t}}$$

$$\therefore \underline{Y} \sim N_2\left[\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix} \right]$$

H.w Let $\underline{X} \sim N_3\left[\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 \\ & 4 & 2 \\ & & 6 \end{pmatrix} \right]$.

Find the distribution of $Y_1 = 2X_1 + 4X_3 +$, $Y_2 = X_1 - 5X_3$ using the charac. fun.

Theorem 8 Suppose that $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ with $\det \Sigma > 0$. Then

$$(\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\underline{\mu}) \sim \chi^2_{(p)}$$

where p the dimension of $\underline{X}$.

proof

Set $\underline{Y} = \Sigma^{-\frac{1}{2}} (\underline{X}-\underline{\mu})$. Then

$$E(\underline{Y}) = \underline{0} \text{ and } var(\underline{Y}) = \Sigma^{-\frac{1}{2}} \Sigma \Sigma^{-\frac{1}{2}}$$

$$= \Sigma^{-\frac{1}{2}} \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} \Sigma^{-\frac{1}{2}} = I$$

$\therefore$ $\underline{Y} \sim N_p(\underline{0}, I)$, and it is followos:

$$(\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\underline{\mu}) = (\underline{X}-\underline{\mu})' \Sigma^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}} (\underline{X}-\underline{\mu})$$

$$= [\Sigma^{-\frac{1}{2}} (\underline{X}-\underline{\mu})]' [\Sigma^{-\frac{1}{2}} (\underline{X}-\underline{\mu})]$$

$$= \underline{Y}' \underline{Y} \sim \chi^2_{(p)}.$$