(ii) Similarly for the multivariate case we have $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ ~~them~~ then the Sample mean $\bar{\underline{X}} = \hat{\underline{\mu}}$

$$\bar{\underline{X}} = \hat{\underline{\mu}} = \frac{1}{n} \sum_{\alpha=1}^{n} X_{i\alpha} \sim N_p\left(\mu, \frac{1}{n} \Sigma\right), \forall i = 1, 2, \dots, p$$

To test.

One-Sample

$H_0: \mu = \mu_0$
$H_1: \mu \neq \mu_0$
((Σ is unknown (one Sample.)))

$\therefore \omega = \{ \underline{\mu} = \underline{\mu}_0, \Sigma_\omega \text{ is unknown} \}$ under H_0

$$\therefore \hat{\omega} = \left\{ \underline{\mu} = \underline{\mu}_0, \hat{\Sigma}_\omega = \frac{\sum_{\alpha=1}^{n} (X_{i\alpha} - \underline{\mu}_0)(X_{i\alpha} - \underline{\mu}_0)'}{n} \right\}$$

$$\Rightarrow L(\hat{\omega}) = (2\pi)^{-\frac{np}{2}} |\hat{\Sigma}_\omega|^{-\frac{n}{2}} e^{-\frac{1}{2} \sum (X_{i\alpha} - \underline{\mu}_0)' \hat{\Sigma}^{-1} (X_{i\alpha} - \underline{\mu}_0)}$$

But $\hat{\Sigma}$ unbiased

$$= \sum_{\alpha=1}^{n} (X_{i\alpha} - \underline{\mu}_0)' \hat{\Sigma}^{-1} (X_{i\alpha} - \underline{\mu}_0)$$

$$= tr\, \hat{\Sigma}^{-1} \sum_{\alpha=1}^{n} (X_{i\alpha} - \underline{\mu}_0)' (X_{i\alpha} - \underline{\mu}_0)'$$

$$= tr\, n \left[\sum_{\alpha=1}^{n} (X_{i\alpha} - \underline{\mu}_0)(X_{i\alpha} - \underline{\mu}_0)' \right]^{-1} \left[\sum_{\alpha=1}^{n} (X_{i\alpha} - \underline{\mu}_0)(X_{i\alpha} - \underline{\mu}_0)' \right]$$

$$= n\, tr\, I_{p \times p}$$

$$= \boxed{np}$$

Then.

$$L(\hat{\omega}) = (2\pi)^{-\frac{np}{2}} |\hat{\Sigma}_\omega|^{-\frac{n}{2}} e^{-\frac{np}{2}} \quad \dots\dots (1)$$

$\Omega = \{-\infty < \underline{\mu} \neq \underline{\mu}_0 < \infty ,\ \Sigma_{\Omega} \text{ is unknown}\}$ under H_1

$\hat{\Omega} = \{ \ \underline{\hat{\mu}} = \underline{\bar{x}} \ , \ \underset{\text{biased}}{\hat{\Sigma}_{\Omega}} = \frac{\sum_{\alpha=1}^{n} (x_\alpha - \bar{x})(x_\alpha - \bar{x})'}{n} \}$

$\hat{\Omega} = (2\pi)^{\frac{-np}{2}} \ |\hat{\Sigma}_{\Omega}| \ e^{-\frac{1}{2} \sum_{\alpha=1}^{n} (x_\alpha - \bar{x})' \hat{\Sigma}_{\Omega}^{-1} (x_\alpha - \bar{x})}$

But

$$= \sum_{\alpha=1}^{n} (x_\alpha - \bar{x})' \hat{\Sigma}_{\Omega}^{-1} (x_\alpha - \bar{x})$$

$$= tr \ \hat{\Sigma}_{\Omega}^{-1} \sum_{\alpha=1}^{n} (x_\alpha - \bar{x})(x_\alpha - \bar{x})'$$

$$= tr \left[n \sum_{\alpha=1}^{n} (x_\alpha - \bar{x})(x_\alpha - \bar{x})' \right]^{-1} \left[\sum_{\alpha=1}^{n} (x_\alpha - \bar{x})(x_\alpha - \bar{x})' \right]$$

$$= tr \ n \ I_{p \times p}$$

$$= n \ tr \ I_{p \times p}$$

$$= np$$

$$\therefore \ L(\hat{\Omega}) = (2\pi)^{\frac{-np}{2}} \ |\hat{\Sigma}_{\Omega}|^{-\frac{n}{2}} \ e^{-\frac{np}{2}} \ \text{-----} \ (2)$$

By using (G. L. R. T)

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

$$= \frac{(2\pi)^{\frac{-np}{2}} \ |\hat{\Sigma}_{\omega}|^{-\frac{n}{2}} \ e^{-\frac{np}{2}}}{(2\pi)^{\frac{-np}{2}} \ |\hat{\Sigma}_{\Omega}|^{-\frac{n}{2}} \ e^{-\frac{np}{2}}}$$

$$\lambda = \frac{|\hat{\Sigma}_{\omega}|^{-\frac{n}{2}}}{|\hat{\Sigma}_{\Omega}|^{-\frac{n}{2}}}$$

$$\lambda = \left\{ \frac{|\hat{\Sigma}_\omega|}{|\hat{\Sigma}_\Omega|} \right\}^{-n/2}$$

تقلب البسط بدل المقام

$$\lambda = \left\{ \frac{|\hat{\Sigma}_\Omega|}{|\hat{\Sigma}_\omega|} \right\}^{\frac{n}{2}}$$

نرفع للقوة ($\frac{2}{n}$) للطرفين

$$\lambda^{\frac{2}{n}} = \frac{|\hat{\Sigma}_\Omega|}{|\hat{\Sigma}_\omega|}$$

$$\lambda^{\frac{2}{n}} = \frac{\left| \sum_{\alpha=1}^{n} (X_\alpha - \bar{X})(X_\alpha - \bar{X})' \right|}{\left| \sum_{\alpha=1}^{n} (X_\alpha - \mu_0)(X_\alpha - \mu_0)' \right|}$$

بعد اضافة وطرح $\bar{x}$.

$$= \frac{\left| \sum_{\alpha=1}^{n} (X_\alpha - \bar{X})(X_\alpha - \bar{X})' \right|}{\left| \sum_{\alpha=1}^{n} (X_\alpha - \bar{X})(X_\alpha - \bar{X})' + n(\bar{\underline{X}} - \mu_0)(\bar{\underline{X}} - \mu_0)' \right|}$$

$$= \frac{|(n-1)S|}{|(n-1)S + n(\bar{X} - \mu_0)(\bar{X} - \mu_0)'|}$$

$$\lambda^{\frac{2}{n}} = \frac{1}{1 + \frac{n(\bar{\underline{X}} - \mu_0)(\bar{\underline{X}} - \mu_0)'}{(n-1)S}} \quad \longleftarrow \quad \text{since } \frac{|a|}{|a+b|} = \frac{1}{1 + \frac{b}{a}}$$

$$\lambda^{\frac{2}{n}} = \frac{1}{1 + \frac{n(\bar{\underline{X}} - \mu_0) S^{-1} (\bar{\underline{X}} - \mu_0)}{(n-1)}}$$

But. $n(\bar{\underline{X}} - \mu_0) S^{-1} (\bar{\underline{X}} - \mu_0) = T^2$

$$\therefore \lambda^{\frac{2}{n}} = \frac{1}{1 + \frac{T^2}{n-1}}$$

where T^2 is Hotelling T^2 test.

Know must find the critical region to test $H_0: \underline{\mu} = \underline{\mu}_0$ (one sample) — إيجاد المنطقة الحرجة في حالة عينة واحدة:

is $\boxed{\lambda \leq \lambda_0}$

$$\lambda^{\frac{2}{n}} \leq \lambda_0^{2/n}$$

$$\frac{1}{1 + \frac{T^2}{n-1}} \leq \lambda_0^{2/n}$$

نقلب الطرفين $1 + \frac{T^2}{n-1} \leq \frac{1}{\lambda_0^{2/n}}$

$$1 + \frac{T^2}{n-1} \leq \lambda_0^{-2/n}$$

$$\frac{T^2}{n-1} \leq \lambda_0^{-2/n} - 1$$

$$T^2 \leq (n-1)\left(\lambda_0^{-2/n} - 1\right)$$

(*) $\boxed{T^2_{cal} \geq T^2_{0\,table}}$ (The critical region one-sample)

where $T^2_{0\,table} = \frac{(n-1)p}{(n-p)} F_{p, n-p, (\alpha)}$

$$T^2_{cal} = n(\underline{\bar{x}} - \underline{\mu}_0)' S^{-1} (\underline{\bar{x}} - \underline{\mu}_0)$$

(iii) To test the null hypothesis
That the mean of one normal population equal to the mean of another normal population with Common unknown Dispersion matrix.

2-Sample

i.e

$$X_\alpha^{(i)} \sim N_p(\underline{\mu}^{(i)}, \Sigma),\quad i=1,2$$

where Σ is unknow
(2-Sample)

To test $H_0: \underline{\mu}^{(1)} = \underline{\mu}^{(2)}$

$$\bar{\underline{X}}^{(1)} \sim N_p(\underline{\mu}^{(1)}, \tfrac{1}{n_1}\Sigma)$$

$$\bar{\underline{X}}^{(2)} \sim N_p(\underline{\mu}^{(2)}, \tfrac{1}{n_2}\Sigma)$$

$$(\bar{\underline{X}}^{(1)} - \bar{\underline{X}}^{(2)}) \sim N_p\left[(\underline{\mu}^{(1)} - \underline{\mu}^{(2)}), (\tfrac{1}{n_1}+\tfrac{1}{n_2})\Sigma\right]$$

$$\therefore \sqrt{\frac{n_1+n_2}{n_1 n_2}}\,(\bar{\underline{X}}^{(1)} - \bar{\underline{X}}^{(2)}) \sim N_p\left[(\underline{\mu}^{(1)} - \underline{\mu}^{(2)}), \Sigma\right]$$

$$\therefore \sqrt{\frac{n_1+n_2}{n_1 n_2}}\,(\bar{\underline{X}}^{(1)} - \bar{\underline{X}}^{(2)}) \sim N_p[0, \Sigma]$$

where

$$S = \frac{\sum_{\alpha=1}^{n}(X_\alpha^{(1)} - \bar{\underline{X}}^{(1)})(X_\alpha^{(1)} - \bar{\underline{X}}^{(1)})' + \sum_{\alpha=1}^{n}(X_\alpha^{(2)} - \bar{\underline{X}}^{(2)})(X_\alpha^{(2)} - \bar{\underline{X}}^{(2)})'}{n_1+n_2-2}$$

There fore

$$T^2 = \frac{n_1 n_2}{n_1+n_2}(\bar{\underline{X}}^{(1)} - \bar{\underline{X}}^{(2)})' S^{-1} (\bar{\underline{X}}^{(1)} - \bar{\underline{X}}^{(2)})$$

The criticle region to test

$H_0: \underline{\mu}^{(1)} = \underline{\mu}^{(2)}$ (Σ unknown) is:

$$T^2 \geq T_0^2 \quad \text{(2- Sample)}$$

Since

$$T_0^2 = \frac{(n_1+n_2-2)p}{n_1+n_2-p-1} F_{p,\, n_1+n_2-p-1}$$

$$T_{Col}^2 = \frac{n_1 n_2}{n_1+n_2} (\bar{\underline{X}}^{(1)} - \bar{\underline{X}}^{(2)})' S^{-1} (\bar{\underline{X}}^{(1)} - \bar{\underline{X}}^{(2)})$$

.1 let $\underline{x} \sim N_2(\underline{\mu}, \Sigma)$, The Sample mean vector $\bar{\underline{x}} = \begin{pmatrix} 55.24 \\ 34.97 \end{pmatrix}$, The Sample Covariance matrix is

$$S = \begin{pmatrix} 210.54 & 126.99 \\ & 119.68 \end{pmatrix}$$, The Sample size

$N = 101$, Test the hypothesis at $\alpha = 0.01$

$$H_0: \underline{\mu} = \underline{\mu}_0 = \begin{bmatrix} 60 \\ 50 \end{bmatrix}$$

Sol

we have Σ know. , $N = 101$ (one sample)

By using Hotelling T^2:

$$T^2 = n(\bar{\underline{x}} - \underline{\mu}_0)' S^{-1} (\bar{\underline{x}} - \underline{\mu}_0)$$

$$= (101)(55.24 \quad 34.97) \begin{pmatrix} 210.54 & 126.99 \\ & 119.68 \end{pmatrix}^{-1} \begin{pmatrix} 55.24 \\ 34.97 \end{pmatrix}$$

$$T^2_{Cal} = 357.43$$

$F_{(2, 99, 0.01)} = 4.83$ from F table

$$\therefore T_0^2 = \frac{(100)(2)}{99}(4.83) = 9.76$$

$$T^2_{Cal} = 357.43 > T_0^2 = 9.76$$

$\therefore$ we reject H_0

i.e. $\underline{\mu} \neq \underline{\mu}_0$

example 2: Let [illegible] where $n_1 = 47$, $n_2 = 48$, the sample means are:

$$\bar{X}^{(1)} = \begin{bmatrix} [illegible] \\ [illegible] \\ [illegible] \\ [illegible] \end{bmatrix}, \quad \bar{X}^{(2)} = \begin{bmatrix} [illegible] \\ [illegible] \\ [illegible] \\ [illegible] \end{bmatrix}$$

and the 4×4 matrix

$$\Sigma = \begin{bmatrix} [illegible] & [illegible] & [illegible] & [illegible] \\ & [illegible] & [illegible] & [illegible] \\ & & [illegible] & [illegible] \\ & & & [illegible] \end{bmatrix}$$

Test the hypothesis: $H_0: \mu_1 = \mu_2$

at $\alpha = 0.05$

2-Sample

Sol

(2 sample)(Σ known) then

$$T^2 = \frac{n_1 n_2}{n_1 + n_2} \left(\bar{X}^{(1)} - \bar{X}^{(2)}\right)' \Sigma^{-1} \left(\bar{X}^{(1)} - \bar{X}^{(2)}\right)$$

$$= \frac{[illegible]}{[illegible]} \begin{bmatrix} [illegible] & [illegible] & [illegible] & [illegible] \end{bmatrix} \begin{bmatrix} [illegible] \end{bmatrix} \begin{bmatrix} [illegible] \end{bmatrix}$$

$$T^2_{cal} = 22.85$$

$$T^2_{table} = \frac{p(n_1 + n_2 - 2)}{n_1 + n_2 - p - 1} F_{[illegible]}$$

$$= \frac{[illegible]}{[illegible]} F_{[illegible]} = \frac{[illegible]}{[illegible]} (5.16) = 5.41$$

$\therefore T^2_{cal} = 22.85 > T^2_{table} = 5.41$

$\therefore$ we reject H_0;

Then $\mu_1 \neq \mu_2$

[illegible]

Note

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Case [2]

[illegible]

$H_0: \mu^{(1)} = \mu^{(2)}$ [illegible]

$H_1: \mu^{(1)} \neq \mu^{(2)}$ [illegible]

(1) [illegible] (Σ_1 and Σ_2 are known)

[illegible]

[illegible]

Let: $Y_\alpha = X_\alpha^{(1)} - X_\alpha^{(2)}$, $\alpha = 1, 2, \ldots, n$

and $\nu = \mu^{(1)} - \mu^{(2)}$

$\Sigma = \Sigma_1 + \Sigma_2$

Therefore will be

$H_0: \nu = 0$

v.s. $H_1: \nu \neq 0$

when Σ is known.

$\bar{Y} \sim N_p(\nu, \frac{1}{n}\Sigma)$

∴ we reject H_0 if $n \bar{Y}' \Sigma^{-1} \bar{Y} \geq \chi^2_{(p)}$ [illegible]

i.e. if $n(\bar{X}^{(1)} - \bar{X}^{(2)})' (\Sigma_1 + \Sigma_2)^{-1} (\bar{X}^{(1)} - \bar{X}^{(2)}) \geq$ [illegible]

[illegible]

(ii) if $n_1 = n_2 = n$ (Σ_1 and Σ_2 are unknown)

$\bar{\underline{X}}^{(1)} \sim N_p(\underline{\mu}^{(1)}, \frac{1}{n_1}\Sigma_1)$, $\bar{\underline{X}}^{(2)} \sim N_p(\mu_2, \frac{1}{n_2}\Sigma_2)$

let $Y_\alpha = X_\alpha^{(1)} - X_\alpha^{(2)} \sim (\nu, \Sigma)$

where $\nu = \bar{\underline{X}}^{(1)} - \bar{\underline{X}}^{(2)}$

$\Sigma = \Sigma_1 + \Sigma_2$

To test $H_0: \nu = 0$, (Σ is unknown)

v.s. $H_1: \nu \neq 0$

$$\therefore T^2 = n\bar{\underline{Y}}' S^{-1} \bar{\underline{Y}} \sim T^2_{(n-1)}$$

$$n(\bar{\underline{X}}^{(1)} - \bar{\underline{X}}^{(2)})' S^{-1} (\bar{\underline{X}}^{(1)} - \bar{\underline{X}}^{(2)}) \sim T^2_{(n-1)}$$

Since $S = \dfrac{\Sigma (Y_\alpha - \bar{Y}_\alpha)(Y_\alpha - \bar{Y}_\alpha)'}{n-1}$

and

$$S = \frac{\Sigma [(X_\alpha^{(1)} - X_\alpha^{(2)}) - (\bar{X}_\alpha^{(1)} - \bar{X}_\alpha^{(2)})][(X_\alpha^{(1)} - X_\alpha^{(2)}) - (\bar{X}_\alpha^{(1)} - \bar{X}_\alpha^{(2)})]'}{n-1}$$

notes

(1) if we found that $\bar{x}_1 = \bar{x}_2$ then we will have the case (iii) with T' and with $d.f = 2n-2$ (in case [illegible])

(2) if $n_1 < n_2$ then $n_1 T'$ [illegible]

For testing $H_0: \mu^{(1)} - \mu^{(2)} = 0$

where

$$S = \frac{\sum_{\alpha=1}^{n_2} \left[X_\alpha^{(2)} - \bar{X}_\alpha^{(2)} - \sqrt{\frac{n_2}{n_1}} \left(X_\alpha^{(1)} - \frac{1}{n_2} \sum [illegible] \right) \right] [illegible]}{(n_2 - 1)}$$

where

$$y = \bar{x}^{(1)} - \bar{x}^{(2)}$$

3. Comparing Two Mean vectors

Assume that we have arandom sample consisting of $\underline{X}_{1i} \sim N_p(\underline{\mu}_1, \Sigma_1)$ $i = 1, 2, \ldots, n_1$ and $\underline{X}_{2i} \sim N_p(\underline{\mu}_2, \Sigma_2)$ $i = 1, 2, \ldots, n_2$. we assume that the two samples are independent and that $\Sigma_1 = \Sigma_2 = \Sigma$, say with Σ unknown.

The test of the equality of means $\underline{\mu}_1$ and $\underline{\mu}_2$ can be formally written as follows:

$$H_0: \underline{\mu}_1 = \underline{\mu}_2 \quad \text{vs} \quad H_1: \underline{\mu}_1 \neq \underline{\mu}_2$$

or

$$H_0: \underline{\mu}_1 - \underline{\mu}_2 = \underline{0} \quad \text{vs} \quad H_1: \underline{\mu}_1 - \underline{\mu}_2 \neq \underline{0}$$

Both samples provide the statistics $\bar{\underline{X}}_1$, $\bar{\underline{X}}_2$, S_1 and S_2.

Let $\delta = \underline{\mu}_1 - \underline{\mu}_2$ and $n = n_1 + n_2$.

we have,

$$(\bar{\underline{X}}_1 - \bar{\underline{X}}_2) \sim N_p\left(\delta, \frac{n}{n_1 n_2}\Sigma\right)$$

and $(n_1 - 1)S_1 + (n_2 - 1)S_2 \sim W_p(n_1 + n_2 - 2, \Sigma)$

Let $S_p = \frac{1}{n_1+n_2-2}\left[(n_1-1)S_1 + (n_2-1)S_2\right]$ is the pool sample covariance matrix.

This lead to a test statistic with Hotelling - T^2 distribution.

$$\frac{n_1 n_2}{n_1+n_2}\left[\{(\underline{\bar{X}}_1-\underline{\bar{X}}_2)-\delta\}' S_p^{-1}\{(\underline{\bar{X}}_1-\underline{\bar{X}}_2)-\delta\}\right] \sim T^2_{(p,\, n_1+n_2-2)}$$

This result can be used to test the null hypothesis

$H_0: \delta = \underline{0}$ or $H_0: \underline{\mu}_1 - \underline{\mu}_2 = \underline{0}$

That is:

$$T^2 = \frac{n_1 n_2}{n_1+n_2}(\underline{\bar{X}}_1-\underline{\bar{X}}_2)' S_p^{-1}(\underline{\bar{X}}_1-\underline{\bar{X}}_2)$$

which is known Hotelling - T^2 statistic.

Then we reject $H_0: \underline{\mu}_1 = \underline{\mu}_2$ if:

$$T^2 \geq T^2_{(\alpha, p, n_1+n_2-2)}$$

If H_0 is true, then

$$\frac{(n_1+n_2-p-1)}{p(n_1+n_2-2)} T^2 \sim F_{(p,\, n_1+n_2-p-1)}$$

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The rejection region of the test is:

$$\frac{(n_1+n_2-p-1)}{p(n_1+n_2-2)} T^2 \geq F_{(\alpha, p, n_1+n_2-2)}$$

Ex: Four psychological test were given (32) men and (32) woman.
If you have the following data:

$$\bar{\underline{x}}_1 = \begin{pmatrix} 15.97 \\ 15.91 \\ 27.19 \\ 22.75 \end{pmatrix}, \quad \bar{\underline{x}}_2 = \begin{pmatrix} 12.34 \\ 13.91 \\ 16.66 \\ 21.94 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 5.192 & 4.549 & 6.522 & 5.250 \\ & 13.18 & 6.760 & 6.266 \\ & & 28.67 & 14.47 \\ & & & 16.65 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 9.136 & 7.549 & 4.864 & 4.151 \\ & 18.60 & 10.22 & 5.446 \\ & & 30.04 & 13.49 \\ & & & 28.00 \end{pmatrix}$$

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Test the hypothesis $H_0: \underline{\mu}_1 = \underline{\mu}_2$ vs.

$H_1: \underline{\mu}_1 \neq \underline{\mu}_2$ at $\alpha = 0.05$

Sol

$$S_p = \frac{1}{n_1+n_2-2}\left[(n_1-1)S_1 + (n_2-1)S_2\right]$$

$$= \frac{1}{32+32-2}\left[(32-1)\begin{pmatrix} 5.192 & 4.545 & 6.522 & 5.250 \\ & 13.18 & 6.760 & 6.266 \\ & & 28.67 & 14.47 \\ & & & [illegible] \end{pmatrix} + (32-1)\begin{pmatrix} 9.136 & 7.549 & 4.864 & 4.151 \\ & 18.60 & 10.22 & 5.446 \\ & & 30.04 & 13.49 \\ & & & 28.00 \end{pmatrix}\right]$$

$$\therefore S_p = \begin{pmatrix} 7.164 & 6.047 & 5.693 & 4.701 \\ & 15.89 & 8.492 & 5.856 \\ & & 29.36 & 13.98 \\ & & & 22.32 \end{pmatrix}$$

$$\therefore T^2 = \frac{n_1 n_2}{n_1+n_2}(\bar{\underline{x}}_1 - \bar{\underline{x}}_2)' S_p^{-1} (\bar{\underline{x}}_1 - \bar{\underline{x}}_2)$$

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$$= \frac{(32)(32)}{32+32} \begin{pmatrix} 15.97 - 12.34 \\ 15.91 - 13.91 \\ 27.19 - 16.66 \\ 22.75 - 21.94 \end{pmatrix}'$$

$$\begin{pmatrix} 7.164 & 6.047 & 5.693 & 4.701 \\ & 15.89 & 8.492 & 5.856 \\ & & 29.36 & 13.98 \\ & & & 22.32 \end{pmatrix}^{-1} \begin{pmatrix} 15.97 - 12.34 \\ 15.91 - 13.91 \\ 27.19 - 16.66 \\ 22.75 - 21.94 \end{pmatrix}$$

$\therefore T^2 = 97.601$

$$F_{cal} = \frac{(n_1+n_2-p-1)}{p(n_1+n_2-2)} T^2 = \frac{(32+32-4-1)}{4(32+32-2)} (97.601) = 23.2$$

$F_{(\alpha, p, n_1+n_2-p-1)} = F_{(0.05, 4, 59)} = 2.335$

Since $F_{cal} = 23.219 > F_{tab} = 2.335$

Therefore, we reject $H_0: \underline{\mu}_1 = \underline{\mu}_2$ and accept $H_1: \underline{\mu}_1 \neq \underline{\mu}_2$

"بسم الله الرحمن الرحيم"

properties of T^2

Case A

a r.s. $X_1, \ldots, X_n$ is drawn from $N_p(\underline{\mu}, \Sigma)$, ($\Sigma$ is unknown), to test

$H_0: \underline{\mu} = \underline{\mu}_0$

v.s. $H_1: \underline{\mu} \neq \underline{\mu}_0$

The test based on the following facts:

① $T^2 = n(\bar{x} - \mu_0)' S^{-1} (\bar{x} - \mu_0) \sim T^2$

where
$$S = \frac{\sum (x_\alpha - \bar{x})(x_\alpha - \bar{x})'}{n-1}$$
where S is unbiased estimate of Σ.

② we can compute T^2 as:
$$T^2 = \frac{|S + n(\bar{x} - \mu)(\bar{x} - \mu)'|}{|S|} - 1$$

③ The dist. of $U = \frac{n-p}{(n-1)(n-p)} T^2 \sim F'_{(p-1, n-p, \alpha)}$

where F' is the non central F with non centrality $n(\mu - \mu_0)' \Sigma^{-1} (\mu - \mu_0)$ if H_0 is true we have Central F

④ The GLR test for $H_0: \mu = \mu_0$ v.s. $H_1: \mu \neq \mu_0$

is $T^2 \geqslant T_0^2$ where T_0^2 in term of F.

Case B let $X_\alpha^{(i)}$, $\alpha = 1, 2, \ldots, n_i$ is a r.s. from $N_p(\mu^{(i)}, \Sigma)$ ...

to test $H_0: \mu = \mu_0$

v.s. $H_1: \mu \neq \mu_0$

The test based on the following facts.

(1) $\bar{X}_i \sim N_p(\mu^{(i)}, \frac{1}{n_i}\Sigma)$, $i = 1, 2$

$\therefore \sqrt{\frac{n_1 n_2}{n_1+n_2}} (\bar{X}^{(i)} - \mu^{(i)}) \sim N_p(0, \Sigma)$

where

$$S = \frac{\sum_{\alpha=1}^{n} (X_\alpha^{(1)} - \bar{X}^{(1)})(X_\alpha^{(1)} - \bar{X}^{(1)})' + \sum_{\alpha=1}^{n} (X_\alpha^{(2)} - \bar{X}^{(2)})(X_\alpha^{(2)} - \bar{X}^{(2)})'}{n_1+n_2-2}$$

There fore. when H_0 is true then

$$\frac{n_1 n_2}{n_1+n_2} (\bar{X}^{(1)} - \bar{X}^{(2)})' S^{-1} (\bar{X}^{(1)} - \bar{X}^{(2)}) \sim T^2_{\ldots}$$

(2) The dist. of

$$u = \frac{n_1+n_2-p-1}{(p-1)(n_1+n_2-2)} T^2 \sim F_{\ldots}$$

(3) The GLR test for $H_0: \mu^{(1)} = \mu^{(2)}$

is $T^2 \geq T_0^2$

where T_0^2 in term of F.

Case C

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والصلاة والسلام على محمد وآل محمد .

<u>Case C</u>: let $X_\alpha^{(i)}$, $\alpha = 1, 2, \dots, n$ is a r.s. from $N_p(\underline{\mu}^{(i)}, \Sigma_i)$, i=

to test:

$H_0: \underline{\mu}^{(1)} = \underline{\mu}^{(2)}$, v.s. $H_1: \underline{\mu}^{(1)} \neq \underline{\mu}^{(2)}$

The test based on the following facts:

① $\bar{x}^{(i)} \sim N_p(\underline{\mu}^{(i)}, \frac{1}{n}\Sigma_i)$, $i = 1, 2$

$x^{(1)} \sim N_p(\underline{\mu}^{(1)}, \Sigma_1)$, $x^{(2)} \sim N_p(\underline{\mu}^{(2)}, \Sigma_2)$

② To test $H_0: \underline{\mu}^{(1)} = \underline{\mu}^{(2)}$ is equavelant to

$H_0: \underline{\mu}^{(1)} - \underline{\mu}^{(2)} = 0$

let $Y_\alpha = X_\alpha^{(1)} - X_\alpha^{(2)}$, $\underline{\mu}_Y = \underline{\mu}^{(1)} - \underline{\mu}^{(2)}$

$\Sigma_Y = \Sigma_1 - \Sigma_2$

$\therefore \bar{Y} \sim N_p(0, \frac{1}{n}\Sigma_Y)$ if H_0 is true.

③ To test $H_0: \mu^{(1)} - \mu^{(2)} = 0$

v.s. $H_1: \mu^{(1)} - \mu^{(2)} \neq 0$

$\because n\bar{y}' S^{-1} \bar{y} \sim T^2_{(n-1)}$

where $S = \frac{\sum (Y_\alpha - \bar{Y})(Y_\alpha - \bar{Y})'}{n-1}$

Simultaneous inferences for means

Case (1) (one- Sample)

let a single Sample of size N has been drawn from $MVN(\underline{\mu}, \Sigma)$, then for any vector $\underset{1\times p}{\underline{a}'} = [a_1\ a_2 \dots a_p]$ to test the hypothesis

$$H_0:\ \underline{a}'\underline{\mu} = \underline{a}'\underline{\mu}_0$$

$$t_{(a)} = \frac{\underline{a}'(\bar{\underline{x}} - \underline{\mu})\sqrt{N}}{\sqrt{\underline{a}' S \underline{a}}} \sim T^2_{\alpha,p,N}$$

$$\Rightarrow t^2_{(a)} = \frac{N(\underline{a}'(\bar{\underline{x}} - \underline{\mu}))^2}{\underline{a}' S \underline{a}} \leq N(\bar{\underline{x}} - \underline{\mu})' S^{-1}(\bar{\underline{x}} - \underline{\mu})$$

and from the dist. of T^2:

$$P\left[t^2_{(a)} \leq T^2_{(\alpha,p,N-p)}\right] = 1-\alpha$$

$$P\left(-T_{(\alpha,p,N-p)} \leq \frac{\underline{a}'\bar{\underline{x}} - \underline{a}'\underline{\mu}}{\sqrt{\frac{\underline{a}' S \underline{a}}{N}}} \leq T_{(\alpha,p,N-p)}\right) = 1-\alpha$$

$$P\left(\underline{a}'\underline{\bar{x}} - \sqrt{\frac{\underline{a}'S\underline{a}}{N}}\, T_{\alpha,p,N-p} \leq \underline{a}'\underline{\mu} < \underline{a}'\underline{\bar{x}} + \sqrt{\frac{\underline{a}'S\underline{a}}{N}}\, T_{\alpha,p,N-p} \right)$$

Since

$$T_{\alpha,p,N-p} = \sqrt{\frac{(n-1)p}{(n-p)} f_{\alpha,p,N-p}}$$

This is the family of the samutaneous confidence intervals of Roy & Bose (1953) with cofficient $(1-\alpha)$ so for a given choice of α we may construct intervals for any and all linear compounds of the means with any hypothesis:

$$H_0: \underline{a}'\underline{\mu} = \underline{a}'\underline{\mu}_0$$

بسم الله الرحمن الرحيم

ex.) A sample of size $n=101$ with $\bar{X} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} 55.24 \\ 34.97 \end{pmatrix}$

and $S = \begin{pmatrix} 210.54 & 126.99 \\ & 119.68 \end{pmatrix}$

Test the hypothesis.

(1) $H_0: \mu_1 = 60$ with $\alpha = 0.01$

sol

$H_0: a'\mu = \mu_1$

$$a'\mu = (1 \ 0)\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = (1 \ 0)\begin{pmatrix} 60 \\ 50 \end{pmatrix} = 60$$

we find

$$\sqrt{\frac{a'Sa}{N}} = \left[\frac{1}{101}(1 \ 0)\begin{pmatrix} 210.54 & 126.99 \\ & 119.68 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^{1/2} = 1.4439$$

$$T_{0.01, 2, 99} = \sqrt{\frac{(101-1)(2)}{(101-2)} F_{0.01, 2, 99}} = \sqrt{\frac{200}{99}(4.79)} = 3.11$$

Then the C.I.

$$Pr\left(55.24 - 1.4439(3.11) \leq \mu_1 \leq 55.24 + 1.4439(3.11)\right) = 0.99$$

$$Pr\left(50.73 \leq \mu_1 \leq 59.75\right) = 0.99$$

$\therefore$ we reject H_0

$H_0: \mu_1 = 60$

H.W:

Test

① $H_0: \mu_2 = 50$

② $H_0: \mu_1 + \mu_2 = 110$

③ $H_0: \mu_1 - \mu_2 = 10$

from the example above