المحاضرة الأولى
2:45
الاثنين 14/ 3/ 2017

8. outer measure

توسع المجال

an outer measure on Ω is a nonnegative extended real valued set fun. λ on the classes of all subsets of Ω satisfying

(a) $\lambda(\emptyset) = 0$

تتزايد بزيادة او بتناقص بنقص

(b) $A \subset B$ implies $\lambda(A) \leq \lambda(B)$ (monotonicity)

(c) $\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \lambda(A_n)$ (countable sub additi

outer mea

ليس شرطا ان تكون غير مشتركة disjoint

example.
The set fun. μ^* of lemma (3) is an outer measure on Ω Check

Def. 8 Outer measure

[illegible]

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Theorem 4

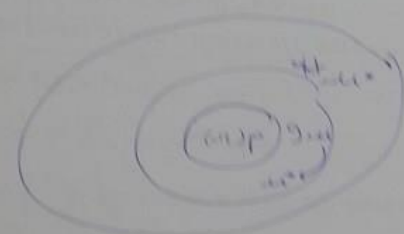
under the hypothesis of lemma (3)

let $\mathcal{H} = \{H \subset \Omega : \mu^*(H) + \mu^*(H^c) = 1\}$

$\mathcal{F}_0 \subset \mathcal{H}$ by lemma (a) and (b) and

$\mathcal{H} = \{H \subset \Omega : \mu^*(H) + \mu^*(H^c) \le 1\}$ by lemma (3)(b)

then $\mathcal{H}$ is a σ-field and μ^* is a prob. measure on $\mathcal{H}$



<u>rem 5</u> A finite measure on a field F_0 can be extended to a measure on $\sigma(F_0)$

<u>proof</u>

Nothing is lost by considering a prob. measure.
the result follows from lemmas (1–3) and theorem (4)
if we observe that $F_0 \subset \mathcal{G} \subset \mathcal{H}$ hence $\sigma(F_0) \subset \mathcal{H}$
Thus μ^* restrected to $\sigma(F_0)$ is the desired extension.

in fact there is very little difference between $\sigma(F_0)$ and $\mathcal{H}$.

if $B \in \mathcal{H}$ then B can be written as $A \cup N$, where $A \in \sigma(F_0)$ and N is a subset of a set $M \in \sigma(F_0)$ with $\mu^*(M) = 0$

to istablish this we introduce the idea of completion of a measure space.

$$F_0 = \{\phi, A, A^c, \Omega\}$$
$$F_1 = \{\text{set of all subset of } \Omega\}$$
$$F_0 \subset F_1$$

الاعداد النسبية $Q \subset R$

$\mu(0, \sqrt{2}) = \sqrt{2}$ ... $\sqrt{2} = 1.4 \dots \infty$? خطأ التقريب

2.23

Def. 9

a measure μ on a σ- field F is said to be complete iff whenever $A \in F$ and $\mu(A) = 0$ we have $B \in F$ for all $B \subset A$, in the theorem(4) μ^* on H is complete for if $B \in A \in H$ where $\mu^*(A) = 0$, then

$$\mu^*(B) + \mu^*(B^c) \leq \mu^*(A) + \mu^*(B^c) = \mu^*(B^c)$$

Thus $\Rightarrow \quad B \in H$

1.10 The completion of a measure space (Ω, F, μ) is defined as follows:

let F_μ be the class of sets $A \cup N$

مداها يتجاوز

where A ranges over $\boxed{F}$ and $\boxed{N}$ over all subsets of sets of measure 0 in F.

zero

Now F_μ is a σ - field including F for it is closed under countable union we extend $\boxed{\mu}$ to $\boxed{F_\mu}$ by setting

$$\mu(A \cup N) = \mu(A)$$

The measure space (Ω, F, μ) is called the Completion of (Ω, F, μ).

and $\boxed{F_\mu}$ is the Completion of $\boxed{F}$ relative to $\boxed{\mu}$

.10

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Theorem (7) $\boxed{\text{Monotone class theorem}}$

let F_0 be a field of subsets of Ω and $\mathcal{C}$ a class of subsets of Ω that is monotone.

أي (if $A_n \in \mathcal{C}$ and $A_n \uparrow A$ or $A_n \downarrow A$ then $A \in \mathcal{C}$)

if $\mathcal{C} \subset F_0$ then $\mathcal{C} \supset \sigma(F_0)$

(the minimal σ-field over F_0)

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Theorem 8 (The fundamental extension theorem)
(or Caratheodory extension theorem)

Let μ be a measure on the field F_0 of subsets of Ω and assume that μ is σ-finite on F_0

تحلل

(so that Ω can be decomposed as $\bigcup_{n=1}^{\infty} A_n$ where $A_n \in F_0$ and $\mu(A_n) < \infty \ \forall n$)

شرط يـ σ-field

then μ has a unique extension to measure on the minimal σ-field F over F_0

Theorem 9 (Approximation theorem)

let (Ω, F, μ) be a measure space and let F_0 be a field of subsets of Ω such that $\sigma(F_0) = F$ assume that μ is σ- finite on on F_0 and

let $\epsilon > 0$ be given if $A \in F$ and $\mu(A) < \infty$

there is a set $B \in F_0$ such that

$$\mu(A \triangle B) < \epsilon$$

فضلة

A فضلة B

مساحة

<u>example</u> let Ω be the rationals, F_0 the field of finite disjoint union of right-semiclosed intervals

$(a,b] = \{w \in \Omega ; a < w \leq b\}$, a,b rationals

[counting $(0,\infty)$ and Ω itself as right-semiclosed intervals]

let $F = \sigma(F_0)$ then

(a) F consist of all subsets of Ω

(b) if $\mu(A)$ is the number of points in A ((μ is counting measure) then μ is $\boxed{\sigma\text{-finite}}$ on F but not on F_0

(c) There are sets $A \in F$ of finite measure that can not be approximated by sets in F_0

i.e There is no sequence $A_n \in F_0$ with $\mu(A \Delta A_n) \to 0$

(d) if $\lambda = 2\mu$, then $\lambda = \mu$ on F_0 but not on F.

Solution

(a) we have $[x] = \bigcap_{n=1}^{\infty} (x - \frac{1}{n}, x)$

and therefore all singleton are in F (since Ω is countable

نعم / لاننا نستطيع كتابة

$\{x\} = \bigcap_{n=1}^{\infty} (x - \frac{1}{n}, x)$

وعليه فان كل singleton موجود في F.

(b) since Ω is countable union of singletons

μ is σ-finite on F.

But every nonempty set in F_0 then has infinite measure so μ is not σ-finite on F_0

نعم / وذلك لان Ω عبارة عن countable union of singletons
اتحاد مفردات منفصلة معدودة

وان μ هو σ-finit في F.

لكن كل المجموعات الغير خالية هي في F_0

وعليه فان لها قياس غير محدد infinite measure

(c) if A is any finite non empty subset of Ω then $\mu(A \Delta B) = \infty$ for all nonempty $B \in F_0$, since any non empty set in F_0 must contains infinitely many point not in A.

اذا كانت A هي مجموعة جزئية محددة وغير خالية من Ω
فانه لكل مجموعة غير خالية B وتنتمي الى F_0 تكون
$\mu(A \Delta B) = \infty$ العدد لنقاط $(A-B)$ هي عدد لانهائي
طالما كل مجموعة غير خالية في F_0 يجب ان تكون من عدد غير محدود من النقاط
غير الموجودة في A

(d) since $\mu(\{x\}) = 1$, $\lambda(\{x\}) = 2$

$\Rightarrow \mu \neq \lambda$ on F.

But $\mu(A) = \infty = \lambda(A)$ on F_0
(except for $A = \emptyset$)

F تحتوي على جميع المجاميع من نوع singleton وعليه فان

$\mu\{x_i\} = 1$, $\forall i$
$\lambda = 2\mu\{x_i\} = 2$, $\forall i$

$\Rightarrow \mu \neq \lambda$ in F
$\mu(A) = \infty = \lambda(A)$
$\Rightarrow \mu = \lambda$ in F_0

Note

كارثيودوري

∴ Both the approximation theorem and
the cartheodory extention theorem
fail in this case.

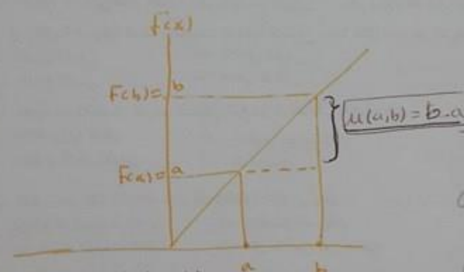
4 Lebesgue - Stieltjes (L-S) Measure and Dist. Fun.

we are going to contruct a large class of Measure on the Borel sets of R

The measure μ defined on right semiclosed interval $(a,b]$ as

$$\mu(a,b] = b-a \quad , \text{ will be extended to the measure}$$

$$\mu(a,b] = F(b) - F(a) \text{ , where } F \text{ is a dist. fun.}$$



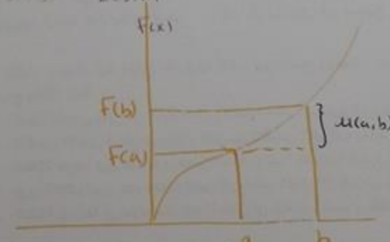
the lenght measure
L.S.m

Lebesgue-stieltjes

(1) $\Omega = R$

(2) measure on $B(R)$
(Borel set $(a,b]$)

(3) $\mu(I)$
interval



The dist. function

Def. 11: A lebesgue - stieltjes measure on R is a measure μ on $\beta(R)$ such that $\mu(I) < \infty$ for each bo
interval I

Real number

rational number — الاعداد النسبية

irrational number — الاعداد غير النسبية

lebesgue – stieltjes Measure

(L. S. M)

[illegible]

rational no.

① $\Omega = R$

② $\beta(R)$

③ $I(a,b]$ — semiclosed [illegible]

④ $\mu(I) = \mu(a,b] = F(b) - F(a)$

[illegible] L.S.M

F is a dist. fun.

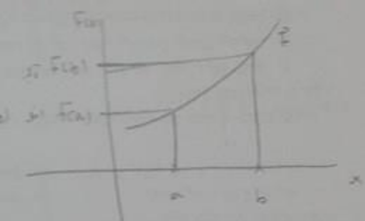
c.d.f.

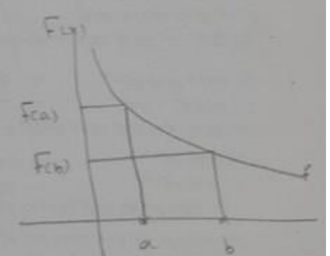
Def. 12

A distribution fun. on R is a map $F : R \to R$
That is increasing [$a < b$ implies $F(a) < F(b)$]

map is a dist. fun on R.

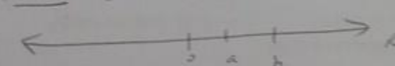
If F increasing $a < b$
$\Rightarrow F(a) \leq F(b)$



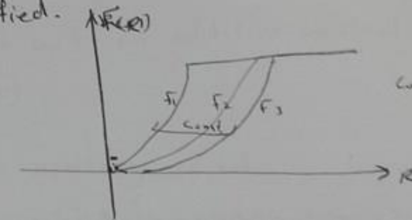


if F decreasing $a < b$
$\Rightarrow F(a) \geq F(b)$

ودالة الاداء تتعلق بخط الاعداد الحقيقية (متزايدة) R.



K (9) Two distribution function that difference by a constant are identified.



Continuous

دالة التوزيع c.d.F اذا تغيرت بالثابت فقط .. const هي متشابهة identified

Discrete

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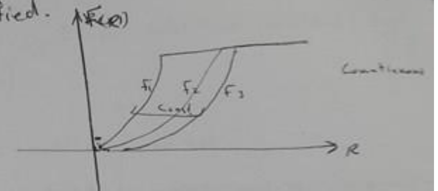

دالة التوزيع c.d.f اذا تغيرت بالثابت فقط .. احده عن const ~~...~~ متشابهة identified

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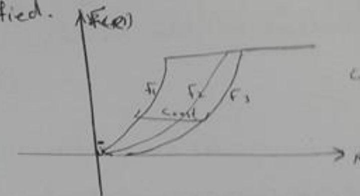


Continuous

identified ~~[illegible]~~ متشابهة بس const .. دالة التوزيع c.d.f اذا تغيرت عن الثانية فقط

Discrete

Theorem (10)

let μ be a L.S. measure on R.
let $F : R \to R$ be defined up to an additive constant, by

$$F(b) - F(a) = \mu(a,b]$$

for example fix $F(0)$ arbitrary and set.

$$F(x) - F(0) = \mu(0,x] \quad , x > 0 \;]$$

$$F(0) - F(x) = \mu(x,0] \quad , x < 0 \;]$$

Then F is a dist. Fun.

F is a map from R to R.

proof (1) if $a < b$

then $F(b) - F(a) = \mu(a,b] \geq 0$

then $F(a) \leq F(b)$

(2) if $\{x_n\}$ is a sequence of points such that

$$x_1 > x_2 > \cdots > \cdots > \to x_\infty$$

then $F(x_n) - F(x) = \mu(x, x_n] \longrightarrow 0$ as $n \to$

$\therefore \lim\limits_{n \to \infty} F(x_n) = F(x)$

$\therefore F(x)$ is right Continuous

For الدالة متصلة من اليمين

i.e F is increasing fun. from R to R and right Continuous
hence it is a distribution fun.

Theorem → Lemma → Remark

Note

let f be adist. on R extend F to amap from $\bar{R}$ to $\bar{R}$ By defining

$$F(\infty) = \lim_{x \to \infty} F(x)$$

$$F(-\infty) = \lim_{x \to -\infty} F(x)$$

((The limits exists by monotonicity))

12.25

Define $\mu(a,b] = F(b) - F(a)$, $\forall\ a,b \in \bar{\mathbb{R}}$, $a<b$

and $\mu[-\infty,b] = \mu(-\infty,b] = F(b) - F(-\infty)$ [illegible]

Then μ is defined on all right semiclosed interval of $\bar{\mathbb{R}}$

[Counting $[-\infty,b]$ as right ~~[illegible]~~ semiclosed]

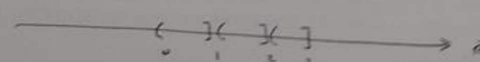
if $I_1, \dots, I_k$ are disjoint right-semiclosed intervals of $\bar{\mathbb{R}}$

we defined $\boxed{\mu\left(\bigcup_{j=1}^{k} I_j\right) = \sum_{j=1}^{k} \mu(I_j)}$

Thus:

μ is extended to the field $F_0(\bar{\mathbb{R}})$ of finite disjoint union of right semiclosed intervals of $\bar{R}$.

and μ is finitely additive of F_0.



$\mu\left(\bigcup_{j=1}^{n} I_j\right) = \sum_{j=1}^{n} \mu(I_j)$

$\bar{\mathbb{R}}$ [illegible]
extended $\mathbb{R}$

[illegible] disjoint

lemma (4)

or: The set fun. μ is Countably additive on F_0 on $\bar{R}$ $F_0(\bar{R})$

proof