Example. let F be the dist. fun. on $\mathbb{R}$ given by

$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1+x & \text{if } -1 \le x < 0 \\ 2+x^2 & \text{if } 0 \le x < 2 \\ 9 & \text{if } x \geq 2 \end{cases}$$

هو ليست prob.

لأن المجموع لا يساوي واحد بل $\underline{9}$

if μ is L.S.M. corresponding to F (as above)

Compute the measure of

ⓐ $\{2\}$, ⓑ $[-\frac{1}{2}, 3)$, ⓒ $(-1,0] \cup (1,2)$, ⓓ $[0,\frac{1}{2}) \cup (1,2]$

ⓔ $\{x : |x| + 2x^2 > 1\}$

Solution

$\underline{p(x \geq 2) - p(\;x < 2)}$

$\mu(x \geq 2) = 9$

$\mu(0 \le x < 2) = 2 + x^2$

ⓐ $\mu(\{2\}) = \mu[2,2] = F(2) - F(\bar{2})$

$= 9 - [2 + (2)^2]$

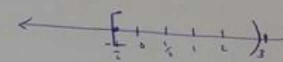
$= 9 - [2+4]$

$= 9 - 6$

$= \boxed{3}$

$\therefore \mu(\{2\}) = 3$

(b) $\mu[-\frac{1}{2}, 3) = \overset{\text{limit}}{F(\bar{3})} - \overset{\text{limit}}{F(-\bar{\frac{1}{2}})}$



$= (x \geq 2) \quad (-1 \leq x < 0)$

$= 9 - [1 + (-\frac{1}{2})]$

$= 9 - [1 - \frac{1}{2}]$

$= 9 - \frac{1}{2}$

$= 8\frac{1}{2} = 8.5$

(c) $\mu(-1, 0] = F(0) - F(-1)$

$= (0 \leq x < 2) - (-1 \leq x < 0)$

$= (2 + (0)^2) - (1 + (-1))$

$= 2 - 0$

$= 2$

$\mu(1, 2) = \overset{\text{limit}}{F(\bar{2})} - F(1)$

$= (0 \leq x < 2) - (0 \leq x < 2)$

$= (2 + (2)^2) - (2 + (1)^2)$

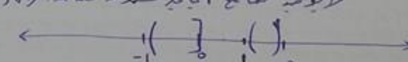
$= (2 + 4) - 3$

$= 6 - 3$

$= 3$

لا يوجد تقاطع أي لا يوجد مشتركة فتأخذ الاتحاد لا نلاحظ مجموع هذه الفترة

or Know.



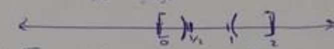
(c) $\mu(-1, 0] \cup \mu(1, 2) = \mu(-1, 0] + \mu(1, 2)$

$= [F(0) - F(-1)] + [F(\bar{2}) - F(1)]$

$= 2 + 3$

$= 5$

لا يوجد تقاطع



(d) $\mu[0, \frac{1}{2}) \cup (1, 2] = \mu[0, \frac{1}{2}) + \mu(1, 2]$

$= [F(\frac{\bar{1}}{2}) - F(\bar{0})] + [F(2) - F(1)]$

$= [(0 \le x < 2) - (-1 \le x < 0)] + [9 - (2 + (1)^2])$

$= [(2 + (\frac{1}{2})^2) - (1 + 0)] + [9 - 3]$

$= (2 + \frac{1}{4} - 1) + 6$

$= 1\frac{1}{4} + 6$

$= 7\frac{1}{4}$

$= 7.25$

(e) $\mu\{x : |x| + 2x^2 > 1\}$

Sol.

we have

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

then

if $x \geq 0$

$x + 2x^2 - 1 \geq 0$

$2x^2 + x - 1 > 0$

$(2x - 1)(x + 1) > 0$

either

$2x - 1 \gtrless 0$ or $x + 1 \gtrless 0$

$2x \gtrless 1$ $\quad$ $x \gtrless -1$

$x \gtrless \frac{1}{2}$ $\quad$ وهذا نهمله

then $2x-1 \geq 0$

$2x \geq 1$

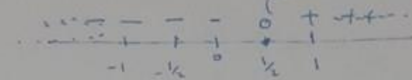
$x \geq \frac{1}{2}$

لإعطاء النقطة

توضيح

$2x-1 \geq 0$

النقطة



$2x-1 \geq 0$

إذا $x = \frac{1}{2}$

$2(\frac{1}{2})-1 = 1-1 = 0$

$x = 0$

$0-1=-1$

$x = -\frac{1}{2}$

$2(-\frac{1}{2})-1 = -2$

$x=-1$

$2(-1)-1=-3$

$x=1$

$2(1)-1=1$

x	$2x-1$	الإشارة
$\frac{1}{2}$	0	0
0	-1	−
$-\frac{1}{2}$	-2	−
-1	-3	−

هكذا نعتبر بأنه النقطة

$(\frac{1}{2}, \infty)$

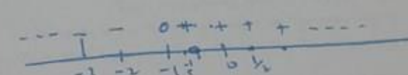
لأنه

$2x-1 \geq 0$

or $x+1 \geq 0$

$x \geq -1$

أما حالة



$x+1 \geq 0$

x	$x+1$	الإشارة
0	1	+1
$\frac{1}{2}$	$1\frac{1}{2}$	+
$-\frac{1}{2}$	$\frac{1}{2}$	+
-1	0	0
-2	-1	−
-3	-2	−

$(-1, \infty)$

if $x \geq 0$

$x + 2x^2 - 1 \geq 0$

$2x^2 + x - 1 \geq 0$

$(2x-1)(x+1) \geq 0$

if $x \geq 0$ عالمنا نريد

فأ

$\left(\frac{1}{2}, \infty\right]$

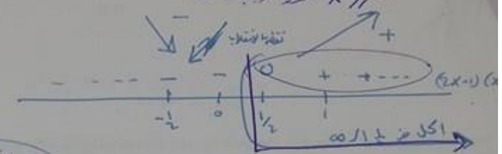
على أن $\frac{1}{2}$ ليست من الفترة المحددة

إذا

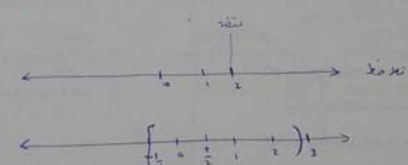
$\left(\frac{1}{2}, \infty\right)$

الآن لنحدد النقطة ص

$x \geq 0$ فقط الموجب نأخذ.



النقطة	$(2x-1)(x+1)$	الإشارة
1	$(2(1)-1)(1+1)$ $(2-1)(1) = (1)(1)$ $= 1$	$+$
$\frac{1}{2}$	$\left(2\left(\frac{1}{2}\right)-1\right)\left(\frac{1}{2}+1\right)$ $(1-1)\left(1\frac{1}{2}\right)$ $0\left(1\frac{1}{2}\right) = \boxed{0}$	0
0	$(2(0)-1)(0+1)$ $(-1)(1) = -1$	$-$
$-\frac{1}{2}$	$\left(2\left(-\frac{1}{2}\right)-1\right)\left(-\frac{1}{2}+1\right)$ $(-2)\left(\frac{1}{2}\right)$ $= -1$	$-$



$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ 2.15

if $x < 0$

$-x + 2x^2 - 1 > 0$

$2x^2 - x - 1 > 0$

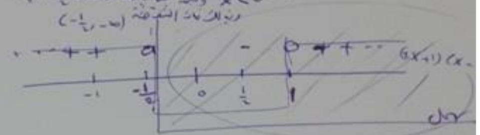
$(2x+1)(x-1) > 0$

either $2x+1 > 0$

or $x - 1 > 0$

$-\frac{1}{2}$ او 1

$x<0$ $(-\frac{1}{2}, -\infty)$



x	$(2x+1)(x-1)$	الإشارة
1	$(2(1)+1)(1-1) = 0$	0
$\frac{1}{2}$	$(2(\frac{1}{2})+1)(\frac{1}{2}-1)$ $(1+1)(-\frac{1}{2})$ $(2)(-\frac{1}{2}) = -1$	−
0	$(2(0)+1)(0-1)$ $(1)(-1) = -1$	−
$-\frac{1}{2}$	$(2(-\frac{1}{2})+1)(-\frac{1}{2}-1)$ $(0)(-1\frac{1}{2})$	0
−1	$(2(-1)+1)(-1-1)$ $(-2+1)(-2)$ $(-1)(-2) = +2$	+
−2	$(2(-2)+1)(-2-1)$ $(-4+1)(-3)$ $(-3)(-3) = +9$	+
2	$(2(2)+1)(2-1)$ $(4+1)(1)$ $(5)(1) = 5$	+
3	$(2(3)+1)(3-1)$ $(7)(2) = 14$	+

$\therefore (-\infty, -\frac{1}{2})$

(e)

$\mu\{x : |x| + 2x^2 > 1\} = \mu\{x : x + 2x^2 > 1\} \cup \mu\{x : -x + 2x^2 > 1\}$

$= \mu(\frac{1}{2}, \infty) \cup \mu(-\infty, -\frac{1}{2})$

$= \mu(\frac{1}{2}, \infty) + \mu(-\infty, -\frac{1}{2})$

$= [F(\infty) - F(\frac{1}{2})] + [F(-\frac{1}{2}) - F(-\infty)]$

$= 9 - (2 + (\frac{1}{2})^2) + [1 + -\frac{1}{2} - 0]$

$9 - 2\frac{1}{4} + \frac{1}{2} = 9 - 1.75 = 7.25$

$= \boxed{7.25}$

Remark (10)

if $\boxed{\mu}$ is finite then $\boxed{F}$ is bounded since $\boxed{F}$ may always be adjusted by additive constant. nothing is lost in this case if we set $\boxed{F(-\infty) = 0}$

Note

if $f : R \longrightarrow R$, $\boxed{f \geq 0}$ and $\boxed{f}$ is integrable (Riemann for now) on any finite interval then if we fix $\boxed{F(0)}$ arbitiary and define

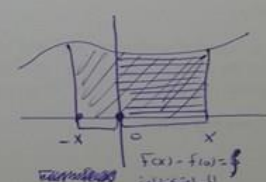
$$F(x) - F(0) = \int_0^x f(t)\,dt \quad , \qquad x > 0$$

and

$$F(0) - F(x) = \int_x^0 f(t)\,dt \quad , \qquad x < 0$$

Then $\boxed{F}$ is a (continuous) dist. Fun. and this gives rise
تعطي وجوداً
ترتفع
to a L.S. μ

$$\mu\,(a,b] = \int_a^b f(x)\,dx$$



(*)

$$\left[\begin{array}{l} \text{if } f(x) = 1, \ \forall x \\ \text{and } F(x) = x \\ \text{then } \mu(a, b] = b - a \end{array}\right]$$

The set fun. μ is called the Lebesgue measure on $\mathcal{B}(R)$
the completion of $\mathcal{B}(R)$ relative to L. M is called the class of (Lebesgue measurable sets).

قابل للقياس

(*) Thus a (Lebesgue measurable sets) is the Union of a Borel set and a subset of a Borel set of Lebesgue measure 0.

(*) The extension of Lebesgue measure to $\overline{\mathcal{B}(R)}$ is called Lebesgue measure also let μ be a L. S. Measure that is concentrated on Countable set $S = \{x_1, x_2, \dots\}$

i.e $\mu(R - S) = 0$

[if (Ω, F, μ) is a measure space and $B \in F$ we say that μ is concentrated on B iff

$\mu(\Omega - B) = 0$]

let

$\Omega = \{1, 2\}$

$F = \{\emptyset, \Omega, \{1\}, \{2\}\}$, $B = \{1\}$

$\mu(B)$

$\mu(\Omega - B) = 0$

$B \in F$

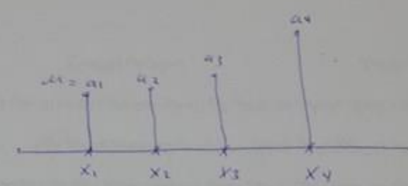
$B \subset \Omega$

in present Case Such that a measure is easily constructed

⊛ if $a_1, a_2, \dots$ are non-negative numbers and $A \subset R$

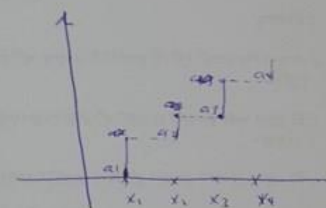
set $\mu(A) = \sum \{a_i : x_i \in A\}$

μ is a measure on all subsets of R not nearly on the Borel sets



$A = \{x_1, x_2, x_3, x_4\}$

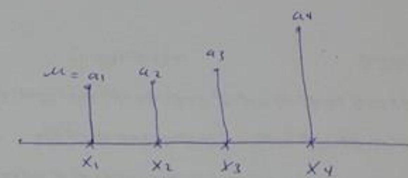
$\mu(A) = a_1 + a_2 + a_3 + a_4$



⊗ if $a_1, a_2, \dots$ are non-negative numbers and $A \subset R$

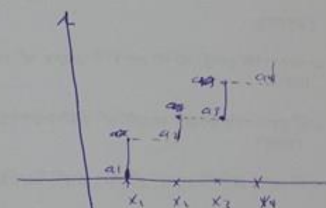
set $\mu(A) = \sum \{a_i : x_i \in A\}$

μ is a measure on all subsets of R not nearly on the Borel sets



$A = \{x_1, x_2, x_3, x_4\}$

$\mu(A) = a_1 + a_2 + a_3 + a_4$



2.45

* if $\mu(I) < \infty$ for each bounded interval I, μ will be a L. S. M. on $B(R)$

if $\sum_{i} a_i < \infty$, μ will be finit measure

* The dist. fun. F corresponding to μ is Continuous on $R - S$

$\sqrt{2} = 1.4142135$

$\Rightarrow \mu(0, \sqrt{2}] = (0, \sqrt{2}) = \sqrt{2} - 0 = ?$

$\sqrt{2} = 1, 1.4, 1.41, 1.414, \ldots$

* if $\mu\{x_n\} = a_n > 0$, F has a jump at x_n of magnitude a_n.

* if $x, y \in S$ and no point of S lies between x and y then F is a constant on $[x, y]$, Fo:

if $x \le b < y$ then $F(b) - F(x) = \mu(x, b] = 0$

if we take S to be the rational numbers,

we have the monotone function $F: R \to R$

عدد غير نسبي

continuous at each irrational number

(point) and discontinuous at each rational point.

Def. 13 a Lebesgue - stieltjes Measure on R^n is a measure μ on $\mathcal{B}(R^n)$ such that $\mu(I) < \infty$ for each bounded interval I.

Def. 14 If $G: R^n \longrightarrow R$ the $\boxed{\text{difference operator}}$ $\underset{b_i,a_i}{\Delta} G(x_1, x_2, \dots, x_n)$ is defined as:

$$\underset{b_i,a_i}{\Delta} G(x_1, x_2, \dots, x_n) = G(x_1, x_2, \dots, x_{i-1}, b_i, x_{i+1}, \dots, x_n) - G(x_1, x_2, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n)$$

$$\underset{b_i,a_i}{\Delta} G(x_1, x_2, \dots, x_n) = G(x_1, \dots, x_{i-1}, b_i, x_{i+1}, \dots, x_n) - G(x_1, x_2, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n)$$

امثلة .

if $x \in R$, $\underset{ba}{\Delta} G(x) = G(b) - G(a)$

if $x_1, x_2 \in R$, $\underset{b_1,a_1}{\Delta} G(x_1, x_2) = G(b_1, x_2) - G(a_1, x_2)$

$\underset{b_2 a_2}{\Delta} G(x_1, x_2) = G(x_1, b_2) - G(x_1, a_2)$

lemma (5)

if $a \le b$ in R^3, i.e, $a_i \le b_i$, $i = 1,2,3$

ⓐ $\mu(a,b] = \underset{b_1 a_1}{\Delta} \underset{b_2 a_2}{\Delta} \underset{b_3 a_3}{\Delta} F(x_1, x_2, x_3)$

ⓑ $\underset{b_1 a_1}{\Delta} \underset{b_2 a_2}{\Delta} \underset{b_3 a_3}{\Delta} F(x_1, x_2, x_3) = F(b_1, b_2, b_3) - F(a_1, b_2, b_3) - F(b_1, a_2, b_3)$

$- F(b_1, b_2, a_3) + F(a_1, a_2, b_3) + F(a_1, b_2, a_3)$

$+ F(b_1, a_2, a_3) - F(a_1, a_2, a_3)$

Where $F(x_1, x_2, x_3) = \mu\{w \in R^3 ; w_1 \le x_1, w_2 \le x_2, w_3 \le x_3\}$

Thus $\mu(a,b] \ne F(b) - F(a)$

proof ⓐ $\underset{b_3 a_3}{\Delta} F(x_1, x_2, x_3) = F(x_1, x_2, b_3) - F(x_1, x_2, a_3)$

$= \mu\{w \in R^3 ; w_1 \le x_1, w_2 \le x_2, w_3 \le b_3\}$

$- \mu\{w \in R^3 ; w_1 \le x_1, w_2 \le x_2, w_3 \le a_3\}$

Since $a_3 \le b_3$

$\underset{b_3 a_3}{\Delta} F(x_1, x_2, x_3) = \mu\{w \in R^3 ; w_1 \le x_1, w_2 \le x_2, a_3 < w_3 \le b_3$

$\therefore \underset{b_2 a_2}{\Delta} \underset{b_3 a_3}{\Delta} F(x_1, x_2, x_3) = \mu\{w \in R^3 ; w_1 \le x_1, a_2 < w_2 \le b_2, a_3 < w_3 \le b_3\}$

$\underset{b_1 a_1}{\Delta} \underset{b_2 a_2}{\Delta} \underset{b_3 a_3}{\Delta} F(x_1, x_2, x_3) = \mu\{w \in R^3 ; a_1 < w_1 \le b_1, a_2 < w_2 \le b_2, a_3 < w_3 \le b_3\}$

(b) H.W.

$$\underset{b_3 a_3}{\Delta} f(x_1, x_2, x_3) = f(x_1, x_2, b_3) - f(x_1, x_2, a_3)$$

$$\underset{b_2 a_2}{\Delta}\, \underset{b_3 a_3}{\Delta} f(x_1, x_2, x_3) = f(x_1, b_1, b_3) - f(x_1, a_2, b_3) - f(x_1, b_2, a_3) + f(x_1, a_2,$$

و هكذا

الا نركز على توزيع

$b_2\, b_3 \quad b_2\, a_3$

$a_2\, b_3 \quad a_2\, a_3$

هكذا البقيه

rk (11)

3.10
ص 51

The extension can be made to n dimensions
<u>i.e</u> μ is a finite Measure on $\mathcal{B}(\bar{R}^n)$ and
hence to Lebesgue-Stieltjes Measure on
R^n or $\bar{R}^n$

The set fun. μ is countably additive on
$\mathcal{F}_0(R^n)$ ~~and~~ if F is a dist. fun. on R^n
and $\mu(a,b] = F(a,b]$, $a,b \in R^n$, $a \leq b$

Then $\exists$ extension of μ to L.S measure on R^n

سؤال امتحان

x.1

let $F_1, F_2, .., F_n$ be dist. fun. on R and define $F(x_1, x_2, .., x_n) = F_1(x_1) \cdots F_n(x_n)$

then F is a dist. fun. on R^n

since $F(a,b] = \prod_{i=1}^{n} [F_i(b_i) - F_i(a_i)]$

in particular, if $F_i(x_i) = x_i$, $i = 1, 2, ..., n$

then each F_i correspondes to L.S.m. on $B(R)$, in this case we have

$F(x_1, .., x_n) = (x_1)(x_2) \cdots (x_n)$

$\therefore \mu(a,b] = F(a,b] = \prod_{i=1}^{n} (b_i - a_i)$

[if $n=3$ we gat volume of cube]

Q// show that if $a \leq b$ where $a, b \in R^3$ (i.e: $a_i \leq b_i$, $i = 1,2,3$) then $\mu(a,b] \neq F(b) - F(a)$.

let f non negative fun. from $R^n \xrightarrow{to} R$

such that:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n < \infty$$

Define.

$$F(x) = \int_{(-\infty, x]} f(t)\, dt.$$

إيجاد
c.d.f.

i.e

$$F(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f(t_1, t_2, \ldots, t_n)\, dt_1 dt_2 \ldots dt_n$$

Thus

$$\Delta_{b_n a_n} F(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_{n-1}} \int_{a_n}^{b_n} f(t_1, \ldots, t_n)\, dt_1 dt_2 \ldots dt_n$$

$$\Rightarrow F[a,b] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(t_1, t_2, \ldots t_n)\, dt_1 dt_2 \ldots dt_n$$

علل: $\therefore F$ is a dist. fun. $\xrightarrow{\text{dist fun}}$ واجب رصاد انها توزيع دالة بتحقق خصائص التوزيع

if μ is L.S.M determined by F

we have

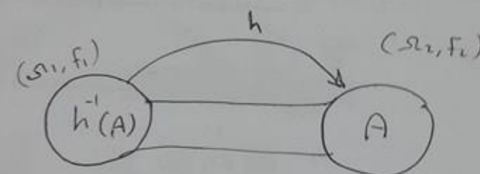
$$\mu(a,b] = \int_{(a,b]} f(x)\, dx$$

Measurable Functions and Integration

Def (15)

if $h : \Omega_1 \longrightarrow \Omega_2$, h is a measurable fun. relative (نسبي) to the fields F_j, $j = 1, 2$

iff $h^{-1}(A) \in F_1$, for each $A \in F_2$



$\therefore$ $h : (\Omega_1, F_1) \longrightarrow (\Omega_2, F_2)$

will mean that $h : \Omega_1 \longrightarrow \Omega_2$ is

measurable relative to F_1 and F_2

[illegible]

it is sufficient that $h^{-1}(A) \in F_1$ for each $A \in \mathcal{C}$, where $\mathcal{C}$ is a class of subsets of Ω_2, such that the minimal σ-field over $\mathcal{C}$ is F_2 for $[A \in F_2 : h^{-1}(A) \in F_1]$ is a σ-field that contains all sets of $\mathcal{C}$. hence

تطابق

concides with F_2

<u>Def.(16)</u> if F is a σ-field of subsets of Ω, (Ω, F) is called <u>measurable sets</u>.

<u>Def.17</u> if (Ω, F) is a measurable space and $h: \Omega \to R^n$ ... h is said to be Borel measurable $[\text{on } (\Omega, F)]$ i... h is measurable

نسبي

relative to the σ-field F an... β the class of Borel sets.

→ if Ω is Borel ~~sets~~ subsets of R^k (or $\bar{R}^k$) and we use the term $\boxed{\text{Borel measurable}}$ we always assume that $F = B$.

هذا يعني بأن الدوال المستمرة كلها قابلة للقياس
بينما الدوال المتقطعة ليست كذلك
وهذا مثال على ذلك.

Remark 12

A continuous map $h: R^k \to R^n$ is Borel Measurable, for if $\mathcal{G}$ is the class of open subsets of R^n the $h^{-1}(A)$ is open hence belongs to $\mathcal{B}(R^k)$ $\forall A \in \mathcal{G}$

if A is a subsets of R that is not a Borel set and I_A is the indicator of A.

i.e $I_A(w) = 1$ for $w \in A$

and $= 0$ if $w \notin A$

then I_A is not Borel Measurable for

$$[w : I_A(w) = 1] = A \notin \mathcal{B}(R)$$

Remark (13)

To show that $h: \Omega \to R$ or $(\bar{R})$ is <u>Borel Measurable</u> it is sufficient to show that $[w: h(w) > c] \in F$ for each c re

For if $\mathcal{C}$ is the class of sets $[x: x > c]$, $c \in R$, then

$$\sigma(\mathcal{C}) = B(R)$$

Similarly $[w: h(w) \geq c]$, $[w: h(w) < c]$ or $[w: h(w) \leq c]$ or equall well $[w: a \leq h(w) \leq b]$ for all real a and b.

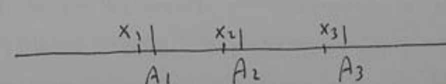
Def. 18

ثابت خلال النقاش.

let (Ω, F) be ameasurable space fixed throughout the discussion. if $h: \Omega \to \bar{R}$, h is said to be simple iff h is Borel measurable and takes on only finitely many distinct values Equivelently h is simple iff it can be written as finitely sum

$$\sum_{i=1}^{r} x_i I_{A_i}$$

where the A_i are disjoint sets in F and I_{A_i} is the inticator of A_i: the x_i need not be disjoint.



Assume that in $\overline{R}$

If $a \in R$,

$$a + \infty = \infty$$
$$a - \infty = -\infty$$
$$\frac{a}{\infty} = \frac{a}{-\infty} = 0$$
$$a \cdot \infty = \infty \text{ if } a > 0$$
$$a \cdot (-\infty) = -\infty \text{ if } a < 0$$
$$0(\infty) = 0(-\infty) = 0$$
$$\infty + \infty = \infty$$
$$-\infty - \infty = -\infty$$

with commulative of additive and multiplicative

Remark 14

The sum differences, product and quotients (القسمة) of simple function are simple $\boxed{\text{check}}$.

ef.19

let h be simple say $h = \sum_{i=1}^{r} x_i I_{A_i}$
where the A_i are disjoint sets in $\mathcal{F}$ we define $\boxed{\text{lebesgue integral}}$ of h with respect to μ written as:

$$\int_{\Omega} h \, d\mu \quad \text{or}$$

or

$$\int_{\Omega} h(\omega) \, d\mu(\omega)$$

or

$$\int_{\Omega} h(\omega) \, \mu(d\omega)$$

لا يظهرا معاً

→ $\int_{\Omega} h \, d\mu = \sum_{i=1}^{r} x_i \, \mu(A_i)$ as long as $+\infty$ and $-\infty$ dont both appear

اذا ظهر معاً ال ∞ ، -∞ فانه ستفرد بدون ان يكون غير معرف

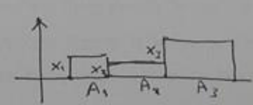
in the sum : if they do we say that the integral does not exist

اذا h لها تقسيم آخر

if h has a different representation, say $\sum_{i=1}^{s} y_i I_{B_i}$

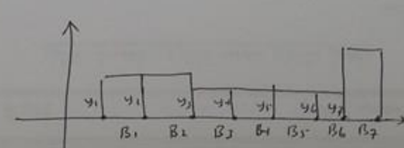
then must.

$$\sum_{i=1}^{r} x_i \mu_{A_i} = \sum_{i=1}^{s} y_i \mu_{B_i}$$



توضيح

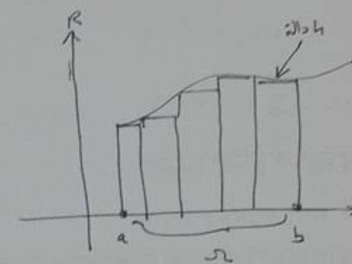
$h = \sum_{i=1}^{3} x_i \mu_{A_i}$, $i = 1,2,3$



h is nonnegative <u>Borel measurable</u> fun. define.

$$\int_{\Omega} h \, d\mu = \sup \left\{ \int_{\Omega} s \, d\mu : s \text{ simple}, \ 0 \leq s \leq h \right\}$$

whis is always exists it may be $+\infty$



if h is arbitrary Borel measurable fun.

let $h^+ = \max\{h, 0\}$

$h^- = \max\{-h, 0\}$

i.e $h^+(\omega) = \begin{cases} h(\omega) & \text{if } h(\omega) \geq 0 \\ 0 & \text{if } h(\omega) < 0 \end{cases}$

$h^-(\omega) = \begin{cases} -h(\omega) & \text{if } h(\omega) < 0 \\ 0 & \text{if } h(\omega) \geq 0 \end{cases}$

h^+, h^- are the positive and the nonnegative parts of h respectively

$|h| = h^+ + h^-$

$h = h^+ - h^-$

where h^+, h^- are Borel measurable

المساواة: تقاطع جزئين

For $\{\omega : h^+(\omega) \in B\} = \{\omega : h(\omega) \geq 0,\ h(\omega) \in B\} \cup \{\omega : h(\omega) < 0,\ 0 \in B\}$

$$\underbrace{\underset{\in F}{h^{-1}[0,\infty)} \cap \underset{\in F}{h^{-1}(B)}}_{\in F} \cup h^{-1}(-\infty, 0) = \emptyset \quad \text{if } 0 \in \emptyset$$

$$= \underset{\in F}{h^{-1}\{0\}}$$

$\geq$

Lebesgue Integral

h is arbitrary B measurable fun

$h^+ = \max\{h, 0\}$, $h^- = \max\{-h, 0\}$

$$h^+(\omega) = \begin{cases} h(\omega) & \text{if } h(\omega) \geq 0 \\ 0 & \text{if } h(\omega) < 0 \end{cases}$$

$$h^-(\omega) = \begin{cases} -h(\omega) & \text{if } h(\omega) \leq 0 \\ 0 & \text{if } h(\omega) > 0 \end{cases}$$

h^+, h^- are the positive and negative parts

$$|h| = h^+ + h^-$$

$$h = h^+ - h^-$$

h^+, h^- are Borel measurable.

h nonnegative B. measurable

$$\int_\Omega h \, d\mu = \sup\left\{ \int_\Omega s \, d\mu : s \text{ simple}, 0 < s < h \right\}$$

وزارة التعليم العالي والبحث العلمي
جامعة المستنصرية

Mark 15

if h_1 and h_2 are Borel measurable then

$\max\{h_1, h_2\}$ and $\min\{h_1, h_2\}$ are Borel measurable

it follows that h^+ and h^- are Borel measurable

$\boxed{\text{prove}}$

if h is Borel measurable, Define

$$\int_\Omega h\,d\mu = \int_\Omega h^+ d\mu - \int_\Omega h^- d\mu \quad , \text{if} \neq +\infty - \infty$$

h is said μ-integrable (integrable) iff $\int_\Omega h\,d\mu < \infty$

i.e $\left(\int_\Omega h^+ d\mu \text{ and } \int_\Omega h^- d\mu \right) < \infty$

if $A \in F$ we define.

$$\int_A h\,d\mu = \int_\Omega h I_A\,d\mu$$

Example

$f(x) = e^{-x} \quad x \geq 0 \quad , \quad A = (0, \infty)$

$F(x) = \int_0^x e^{-u}\, du$ c.d.f. لايجاد

لدينا .

$f(x) = \begin{cases} e^{-x} & , \; x \geq 0 \\ 0 & \;\; x < 0 \end{cases}$ جزء كل

$\Omega = R$ دائماً

$-\infty \to \infty$

density fun.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} e^{-x}(0)\,du + \int_{-\infty}^{\infty} e^{-x}\,du$$

مقرر Beta fun.

eorem 12.

if $h_1, h_2, \ldots,$ are Borel fun. from Ω to R

and $h_n(\omega) \longrightarrow h(\omega)$ $\forall\ \omega \in R$, as $n \longrightarrow \infty$

then h is a Borel measurable

proof

$$\{w : h(\omega) > c\} = \{w : \lim_{n\to\infty} h_n(\omega) > c\}$$

$$= \{w : h_n(\omega) \text{ is eventually} > c + \tfrac{1}{r} \text{ for some } r, r = 1, 2, \ldots\}$$

$$= \bigcup_{r=1}^{\infty} \{w : h_n(\omega) > c + \tfrac{1}{r} \text{ for all but finitely many } n\}$$

اتحاد التقاطعات

$$= \bigcup_{r=1}^{\infty} \liminf \{w : h_n(\omega) > c + \tfrac{1}{r}\}$$

تعريف liminf

$$= \bigcup_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{w : h_k(\omega) > c + \tfrac{1}{r}\} \in F$$

Theorem 13 Approximation theorem

ⓐ a non negative Borel measurable fun. h is the limit of an increasing sequence of non negative finite valued simple fun. h_n.

ⓑ An arbitrary Borel measurable fun. f is the limit of a sequence of finite valued simple fun. f_n.

with $|f_n| \leq |f|$

لأي دالة غير سالبة Borel measurable يمكن اعتبارها غاية لمتتابعة
متصاعدة من دوال بسيطة
ذات قيم حقيقية محددة. ويمكن
التعميم لأي دالة B.M مع
أخذ القيم المطلقة.