ch-2

Further results in Measure and integral theor

introduction.

suppose that f is a dist. fun. on R. and assu

اهمية

that f has $\boxed{a}$ jump of magnitude a_k at points ch-2

Further results in Mea

① introduction.

suppose that F is adist.

that F has $\boxed{a}$ jump of ma

x_k , $k=1,2,\ldots$

let μ_1 be acorrespondin

on $\{x_1, x_2, \ldots\}$ with

$$\mu_1(\{x_k\}) = a_k \quad \forall k$$

and let F_1 be adist. fun.

$G = F - F_1$ is acontinuous

corresponding Lebesgue - ~~stieltjes~~

Satisfies.

$$\lambda\{x\} = 0 \quad \forall x$$

ch-2

Further results in Measure and integral theory.

introduction.

suppose that f is a dist. fun. on R. and assume that f has $\boxed{a}$ jump of أهمية magnitude a_k at points x_k , $k=1,2,\dots$

let μ_1 be a corresponding measure مكثف (مركز) concentrat[ed] on $\{x_1, x_2, \dots\}$ with

$$\mu_1(\{x_k\}) = a_k \quad , \forall k$$

and let F_1 be a dist. fun. corresponding to μ_1 [...] $G = F - F_1$ is a continuous dist. fun. so that [...] corresponding Lebesgue- ~~stieltjes~~ stieltjes measure λ satisfies.

$$\lambda\{x\} = 0 \quad , \forall x$$

نطرح عدم الاستمرارية لـ F

(i.e. we subtract Discontinuities of F)

thus

$$G(x) = \int_{-\infty}^{x} f(t)\,dt \quad , \; x \in R$$

for some non-negative Borel measurable

$$f \; (f = G')$$

hence

$$\lambda(B) = \int_{B} f(x)\,dx \quad \forall B \in \mathcal{B}(R)$$

for one and only one λ.

(the Lebesgue-stieltjes measure determined b

The conjecture of this is not true.

المرافق
العكس

if λ is a measure on $\mathcal{B}(\mathbb{R})$ and $\lambda\{x\}=0, \forall x$

then it is not always true that we can write

$$\lambda(B) = \int_B f(x)\,dx \,, \quad B \in \mathcal{B}(\mathbb{R}),$$

for some non-negative Borel measurable f

it is true if we have in addition $\lambda(B)=0$

whenever $\mu_1(B)=0$

let λ be a countably additive extended real-valued set fun. on the σ-field $\mathcal{F}$ of subsets of Ω

then λ assume a maximum and minimum value

i.e $\exists\ C, D \in \mathcal{F} \ni$

$$\lambda(C) = \sup\{\lambda(A) : A \in \mathcal{F}\} \text{ and}$$
$$\lambda(D) = \inf\{\lambda(A) : A \in \mathcal{F}\}$$

هيلبت ما لان
شرط عدم السالبية
غير موجود.

eorem 2

Jordan–Hahn Decomposition theorem

let λ be a countably additive extended Real valued set fun. on the σ-field f. Define

$$\lambda^+(A) = \sup\{\lambda(B) : B \in f,\ B \subset A\}$$

$$\lambda^-(A) = -\inf\{\lambda(B) : B \in f,\ B \subset A\}$$

then λ^+ and λ^- are measure on f and

$$\lambda = \lambda^+ - \lambda^-$$

llaries

let λ be a countably additive extended real-val…
set fun. on the σ-field $\mathcal{F}$.

(a) the set fun. λ is the difference of measure at least one of which is finite

(b) if λ is finite then λ is bounded

ضروري انو نحط خط تحت السالبة

(c) $\exists\ D \in \mathcal{F} \ni \lambda(A \cap D) \leq 0$

and $\lambda(A \cap D^c) \geq 0 \quad \forall\ A \in \mathcal{F}$

(d) if $D \in \mathcal{F} \ni \lambda(A \cap D) \leq 0$ and $\lambda(A \cap D^c$…

then $\lambda^+(A) = \lambda(A \cap D^c)$

and $\lambda^-(A) = -\lambda(A \cap D) \quad \forall\ A \in \mathcal{F}$

if $E \in F$, $E \neq \Omega \ni \lambda(A \cap E) \leq 0$

and $\lambda(A \cap E^c) \geq 0$, $\forall A \in F$

then $|\lambda| (\Omega \triangle E) = 0$

where $|\lambda| = \lambda^+ + \lambda^-$

2 - Radon Nikodym theorem and Related Results

Def one measure ν is said to be absolutely continuous with respect to another μ if for any set $A \in \mathcal{F}$ and $\mu(A) = 0 \longrightarrow \nu(A) = 0$

written $\nu \ll \mu$

أي أن ν يذهب للصفر مع μ .

الاشارة $\boxed{\ll}$ تعني absolutely continuous

يعني ثابتة تماما .

Theorem 3 . Radon - Nikodym theorem

تعريف

let μ be σ - finite measure and λ a signed measure on the σ - field $\mathcal{F}$ ($\mathcal{F}$ contains subset of Ω)

(i.e. if $A \in \mathcal{F}$ then $|\lambda|(A) = 0$ iff $\lambda(B) = 0$ $\forall B \in \mathcal{F}, B$

Borel

of subset of Ω

assume that λ is absolutely continuous (w.r.t.) μ [$\lambda \ll \mu$] then there is a Borel measurable fun. $g : \Omega \to \bar{\mathbb{R}}$ (دالة) such th

$$\lambda(A) = \int g \, d\mu \quad \forall A \in \mathcal{F}$$

if h is another such fun. then

$h = g$ a.e. $[\mu]$.

Corollaries

under the hypothesis of theorem 3

ⓐ if λ is finite then g is μ-integrable hence finite a.e. $[\mu]$

ⓑ if $|\lambda|$ is σ-finite so that Ω can be expressed as a countable union of sets A_n such that $|\lambda|(A_n)$ is finite (equivalently $\lambda(A_n)$ is finite). then g is finite a.e. $[\mu]$.

ⓒ if λ is a measure, then $g \geq 0$ a.e. $[\mu]$.

The sequence of measurable fun. f_n, $n=1,2,\dots$

تتقارب في كل مكان

(a) is said to be <u>converge everywhere</u> to the measurable fun. g defined on the same measure space $\underline{(\Omega, \mathcal{B}, \mu)}$

if $f_n(\omega) \to g(\omega)$

[means $f_n(\omega)$ converge to $g(\omega)$ for every where

<u>i.e</u>

$$\lim_{n\to\infty} f_n(\omega) = g(\omega) \quad \forall\ \omega \in \Omega$$

تتقارب تقريبا

(b) Converge almost everywhere (w.r.t.) μ if

$f_n(\omega) \to g(\omega)$

except for sets with measure (zero)

if μ is prob. measure we say

$f_n(\omega) \xrightarrow{a.s} g(\omega)$ (almost surely)

تقريبا اكيد.

Converg in pth mean $f_n(\omega) \xrightarrow[\text{Large } P]{L.P} g(\omega)$

iff $\int |f_n(\omega) - g(\omega)|^p \, d\mu \longrightarrow 0$

consider for example μ = prob P (كبير) and $p=2$ (صغير)

$\therefore f_n(\omega) \xrightarrow{L.2} g(\omega)$

بدلاً من μ

$\Rightarrow \int |f_n(\omega) - g(\omega)|^2 \, d\,f(x) \longrightarrow 0$

i.e

$E\left[f_n(\omega) - g(\omega)\right]^2 \longrightarrow 0$ and we say converge in quadratic mean

if $E(X_n - Z)^2 \longrightarrow 0$

then $X_n \xrightarrow{q.m} Z$ (in quadratic mean)

When $p=1$ we say converge in the mean

(d) $f_n(\omega) \xrightarrow{\mu} g(\omega)$ (in measure) iff.

$$\lim_{n\to\infty} \mu \{\omega : |f_n(\omega) - g(\omega)| > \epsilon\} = 0 \quad , \forall \ \epsilon > 0$$

$X_n \xrightarrow{P} Z$ iff.

$$\lim_{n\to\infty} P\{ |X_n - Z| > \epsilon\} = 0 \quad , \forall \ \epsilon > 0$$

and we say X_n converge in prob. to Z

$\left[f_n(\omega) \longrightarrow g(\omega) \right] \Longrightarrow \left[f_n(\omega) \xrightarrow{a.e} g(\omega) \right] \Longrightarrow \left[f_n(\omega) \xrightarrow{\mu} g(\omega) \right]$

Converge everywhere
$\forall \omega \in \Omega$

Converge almost everywhere
except for sets with measure zero

Converge in measure
iff
$\lim \mu[\omega : |f_n(\omega) - g(\omega)| > \epsilon] = 0$
$\forall \epsilon > 0$

$\Longrightarrow \left[f_n(\omega) \xrightarrow{L_1} g(\omega) \right] \Longrightarrow \left[f_n(\omega) \xrightarrow{L^p} g(\omega) \right]$

Converge with mean
iff
$\int |f_n(\omega) - g(\omega)| \, dF(x) \Longrightarrow 0$

converg with p^{th} mean
iff
$\int |f_n(\omega) - g(\omega)|^p \, d\mu \to 0$

Q1 Give an example of a measure μ an a non-negative finite valued Borel measurable fun., g.
such that the measure λ defined by $\lambda(A)=\int_A g\,d\mu$
is not σ- finite

Ash 41, P

Q2/ Give an example in which the conclusion of the Radon - Nikodym theorem but fails in other words,
$\lambda \ll \mu$ but there is no Borel measurable g such
that $\lambda(A)=\int_A g\,d\mu$ for all $A \in \mathcal{F}$. of course
μ can not be σ- finite

Q4/ if $\lambda(A) = \int_A g \, d\mu$, $A \in F$

and g is μ-integrable

we know that λ is finite: in particular, $A = \{w : g(w)$

has finite λ-measure.

show that A has σ-finite , μ-measure , that

it is a countable union of sets of finite measu

Give an example to show that $\mu(A)$

need not be finite.

Sol

تسم (1)

$D = \{w : f(w) < 0\}$ then $\lambda(A \cap D) \leq 0$

$\lambda(A \cap D^c) \geq 0$, $\forall A \in F$

$\lambda^+(A) = \lambda(A \cap D^c) = \int_A f^+ \, d\mu$

since $f^+ = f$ on D^c and $f^+ = 0$ on D

$\lambda^-(A) = \lambda(A \cap D) = \int_A f^- \, d\mu$

since

$f^- = -f$ on D

and $f^- = 0$ on D^c

(ملخص الـ سلامة السابق)

① ν is absolutely continuous with respect to λ ($\nu << \lambda$)

ν is absolutely continuous with respect to λ if for any set $A \in \mathcal{F}$ and $\lambda(A) = 0 \longrightarrow \nu(A) = 0$

اي ينصب الى الصفر قبله.

تنكلا

② Converge a.e[μ]

Converge almost everywhere with respect to μ if $f_n(w) \to g(w)$ except for sets of measure zero

③ Converge in quadratic mean

$f_n(\omega) \xrightarrow{L^p} g(\omega)$ iff $\int |f_n(\omega) - g(\omega)|^p \, d\mu$

Consider $p=2$

$f_n(\omega) \xrightarrow{L^2} g(\omega)$ iff $\int |f_n(\omega) - g(\omega)|^2 \, d\mu \longrightarrow 0$

i.e. $E(f_n(\omega) - g(\omega))^2 \longrightarrow 0$ and we say converge to in quadratic mean

if $E(X_n - Z)^2 \longrightarrow 0$ then $X_n \xrightarrow{qm} Z$ (in quadratic mean)

Consider $p=1$

We say converge in mean.

i.e

A condition is said to be hold almost every where (a.e) with respect to the measure μ (a.e$[\mu]$) iff there is a set $B \in \mathcal{F}$ of μ-measure zero ($\mu(B)=0$) such that the condition hold outside of B.

Monotone Convergence Theorem

let $h_1, h_2, \dots$ form an increasing sequence of nonnegative Borel measurable functions and

let $h(w) = \lim_{n\to\infty} h_n(w)$, $w \in \Omega$ then

$$\int_\Omega h_n \, d\mu \longrightarrow \int_\Omega h \, d\mu$$

[Note that $\int_\Omega h_n \, d\mu$ increases with n]

i.e $0 \le h_n \uparrow h$ implies $\int_\Omega h_n \, d\mu \uparrow \int_\Omega h \, d\mu$

Additive Theorem

let f and g be Borel measurable function and assume $f, g, f+g$ is well define $\left[\int_\Omega f\,d\mu, \int_\Omega g\,d\mu, \int_\Omega (f+g)\,d\mu \text{ exists}\right]$

then

$$\int_\Omega (f+g)\, d\mu = \int_\Omega f \, d\mu + \int_\Omega g \, d\mu$$

Corallarie of Additive theorem

a) if $h_1, h_2, \ldots$ are Borel Measurable function then

$$\int_\Omega \sum_{n=1}^{\infty} h_n \, d\mu = \sum_{n=1}^{\infty} \int_\Omega h_n \, d\mu$$

b) if h is Borel Measurable, h is integrable iff $|h|$ is integrable

c) if g, h are Borel Measurable with $|g| \leq h$, h is integrable then g is integrable.

Extended Monotone Convergence theorem

let $g_1, g_2, \ldots, g, h$ be Borel Measurable functi[ons]

a) if $g_n \geq h$, $\forall n$ where $\int_\Omega h \, d\mu > -\infty$ and $g_n \uparrow g$

then $\int_\Omega g_n \, d\mu \uparrow \int_\Omega g \, d\mu$

b) if $g_n \leq h$ $\forall n$ where $\int_\Omega h \, d\mu < \infty$ and $g_n \downarrow g$

then $\int_\Omega g_n \, d\mu \downarrow \int_\Omega g \, d\mu$

i.e the limit integral under propriate condition is equal to the integral of the limit

Fatou's lemma

let $f_1, f_2, \dots, f$ be Borel measurable functions

(a) if $f_n \geq f$, $\forall n$ where $\int_\Omega f \, d\mu > -\infty$ then

$$\lim_{n\to\infty} \inf \int_\Omega f_n \, d\mu \geq \int_\Omega \lim_{n\to\infty} \inf f_n \, d\mu$$

(b) if $f_n \leq f$, $\forall n$ where $\int_\Omega f \, d\mu < \infty$ then

$$\lim_{n\to\infty} \sup \int_\Omega f_n \, d\mu \leq \int_\Omega \lim_{n\to\infty} \sup f_n \, d\mu$$

Dominated convergence theorem

If $f_1, f_2, \dots, f, g$ are Borel measurable

$|f_n| \leq g$, $\forall n$, where g is μ-integrable ($\int g \, d\mu$

and $f_n \to f$ a.e $[\mu]$

then

f is μ-integrable and

$$\int_\Omega f_n \, d\mu \to \int_\Omega f \, d\mu$$

Corollary

If $f_1, f_2, \ldots, f, g$ are Borel measurable $|f_n| \leq g \ \forall n$ ~~where~~

$|g|^p$ is μ-integrable ($p > 0$ fixed) and $f_n \to f$ a.e $[\mu]$

then

$|f|^p$ is μ-integrable and $\int_{\Omega} |f_n - f|^p \, d\mu \to 0$ as $n \to$

Comparison of Lebesgue and Riemann Integrals

let $[a,b]$ be a bounded closed interval of reals, and let f be bounded real-valued function on $[a,b]$ fixed throughout the discussion $\Rightarrow$ ثابتة خلال النقاش

If $P: a = x_0 < x_1 < \dots < x_n = b$ is a patition of $[a,b]$

We may construct the upper and lower sums of f relative to P as follows

let $M_i = \sup\{f(y) : x_{i-1} < y < x_i\}$, $i = 1, 2, \dots, n$
$m_i = \inf\{f(y) : x_{i-1} < y < x_i\}$, $i = 1, 2, \dots, n$ and

define step functions α and β called the upper and lower functions corresponding to P by

$$\alpha(x) = M_i \quad \text{if } x_{i-1} < x < x_i, \quad i = 1, 2, \dots, n$$
$$\beta(x) = m_i \quad \text{if } x_{i-1} < x < x_i, \quad i = 1, 2, \dots, n$$

The upper and lower sums are given by

$$U(P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

$$L(P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

Since α and β are simple functions we have

$$U(P) = \int_a^b \alpha \, d\mu$$

$$L(P) = \int_a^b \beta \, d\mu$$

Now let $P_1, P_2, \ldots$ be a sequence of partitions of $[a,b]$ such that P_{k+1} is a refinement of P_k for each k and such that

$|P_k|$ (the length of the largest subinterval of P_k) approaches 0 as $k \to \infty$

If α_k and β_k are the upper and lower functions corresponding to P_k then

$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq \cdots \geq f \geq \cdots \geq \beta_k \geq \cdots \geq \beta_2 \geq \beta_1$

thus α_k and β_k approach limit functions α and β.

If $|f|$ is bounded by M then all $|\alpha_k|$ and $|\beta_k|$ are bounded by M as well, and the function is constant at M is integrable on $[a,b]$ with respect to μ. since

$\mu[a,b] = b - a < \infty$.

By the dominated convergence theorem,

$$\lim_{k \to \infty} U(P_k) = \lim_{k \to \infty} \int_a^b \alpha_k \, d\mu = \int_a^b \alpha \, d\mu.$$

and

$$\lim_{k \to \infty} L(P_k) = \lim_{k \to \infty} \int_a^b \beta_k \, d\mu = \int_a^b \beta \, d\mu.$$

If x is not an endpoint of any of the subintervals of the P_k.

f is continuous at x iff $\alpha(x) = f(x) = \beta(x)$

$\lim_{k\to\infty} U(P_k) = \lim_{k\to\infty} L(P_k) =$ a finite number r, independent of the particular sequence of partitions.

f is said to be Riemann integrable on $[a,b]$. and

$r = r_{a,b}(f)$ is said to be the (value of the) Riemann integrable of f on $[a,b]$.

f is Riemann integrable iff $\int_a^b \alpha \, d\mu = \int_a^b \beta \, d\mu = r$

independent of the particular sequence of partitions.

If f is Riemann integrable $r_{a,b}(f) = \int_a^b \alpha \, d\mu = \int_a^b \beta \, d\mu$

Jordan Hahn Decomposition theorem

let λ be a Countably additive extended real valued set function on σ- Field $\mathcal{F}$ Define

$$\lambda^+(A) = \sup \{ \lambda(B) : B \in \mathcal{F}, B \subset A \}$$
$$\lambda^-(A) = \inf \{ \lambda(B) : B \in \mathcal{F}, B \subset A \}$$

then λ^+ and λ^- are measure on $\mathcal{F}$ an $\lambda = \lambda^+ - \lambda^-$

Corollaries

let λ be a Countably additive extended real valued

... on σ-Field $\mathcal{F}$.

λ^+ and λ^- are measure on $\mathcal{F}$ an $\lambda = \lambda^+ - \lambda^-$

Corollaries

let λ be a countably additive extended real valued set function on a σ-Field $\mathcal{F}$.

(a) The set function λ is the difference of two measures at least on of which is finite.

(b) If λ is finite then λ is bounded.

(c) $\exists\, D \in \mathcal{F} \ni \lambda(A \cap D) \leq 0$ and $\lambda(A \cap D^c) \geq 0 \ \forall A \in \mathcal{F}$

(d) If $D \in \mathcal{F} \ni \lambda(A \cap D) \leq 0$ and $\lambda(A \cap D^c) \geq 0 \ \forall A \in \mathcal{F}$ then $\lambda^+(A) = \lambda(A \cap D^c)$ and $\lambda^-(A) = -\lambda(A \cap D) \ \forall A \in \mathcal{F}$

(e) If $E \in \mathcal{F}$, $E \neq D \ni \lambda(A \cap E) \leq 0$ and $\lambda(A \cap E^c) \geq 0 \ \forall A \in \mathcal{F}$ then $|\lambda|(D \Delta E) = 0$ Where $|\lambda| = \lambda^+ + \lambda^-$

Radon-Nikodym theorem

let μ be σ-finite measure and λ a signed measure on the σ-field $\mathcal{F}$

(i.e If $A \in \mathcal{F}$ then $|\lambda|(A)=0$ iff $\lambda(B)=0$ $\forall B \in \mathcal{F}$, $B \subset A$)

of subsets of Ω.

assume that λ is absolutely continuous (w.r.t) μ then there is a Borel measurable function $g: \Omega \to \bar{R}$ such that $\lambda(A) = \int_A g \, d\mu$ $\forall A \in \mathcal{F}$

if h is another such function then $h = g$ a.e $[\mu]$.

if h is another such function then $h=g$ a.e $[\mu]$.

Corollaries

under the hypothesis of theorem (3)

(a) If λ is finite then g is μ-integrable hence finite a.e $[\mu]$.

(b) If $|\lambda|$ is σ-finite so that Ω can be expressed as a countable union of sets A_n such that $|\lambda|(A_n)$ is finite (equivalently $\lambda(A_n)$ is finite)

then g is finite a.e $[\mu]$

(c) If λ is a measure then $g \geq 0$ a.e $[\mu]$.

$[f_n(\omega) \longrightarrow g(\omega)]$ $\not\Rightarrow$ $[f_n(\omega) \xrightarrow{a.e[\mu]} g(\omega)]$ $\not\Rightarrow$ $[f_n(\omega) \xrightarrow{\mu} g(\omega)]$ $\not\Rightarrow$ $[f_n(\omega) \xrightarrow{L_1} g(\omega)]$ $\not\Rightarrow$ $[f_n(\omega) \xrightarrow{L_p} g(\omega)]$

Converge everywhere $\forall \omega \in \Omega$

Converge almost everywhere except for sets of measure

Converge in measure iff $\lim_{n\to\infty} \mu\{\omega : |f_n(\omega) - g(\omega)| > \epsilon\} = 0$ $\forall \epsilon > 0$

Converge with mean iff $\int |f_n(\omega) - g(\omega)| d\mu \to 0$

Converge with pth mean iff $\int |f_n(\omega) - g(\omega)|^p d\mu \to 0$