Ch-3

Random variable and Expectation

Random variable

Expectation

Conditional prob.

Conditional Expectation

prob. of Conditional Expectation.

Ch-3

Random Variable and Expectation

Def r.v.

a random variable X (or extended r.v) on a prob. space $(\Omega, \mathcal{F}, P)$ is a Borel measurable fun. from Ω to R (or $\bar{R}$)

i.e $\quad X : (\Omega, \mathcal{F}) \xrightarrow{\text{image}} (R, \mathcal{B}(R)) \ni$

$$P_X(B) = P\{w : X(\omega) \in B\}, \quad B \in \mathcal{B}(R)$$

$P_X(B)$

متقطع   مستمر

example

let $\Omega = \{HH, HT, TH, TT\}$

and F the set of all subsets of Ω

μ on F is a prob. measure

$\mu = \{HH\} = \mu\{HT\} = \mu\{TH\} = \mu\{TT\} = \frac{1}{4}$

let X be a function defined on Ω such that

$X\{HH\} = 0$, $X\{HT\} = 1$, $X\{TH\} = 1$, $X\{TT\} = 2$

$X : \Omega \to \mathbb{R}$

$X : \Omega \to \overset{\text{image}}{\{0,1,2\}} = \Omega_2$

let F_2 be the set of all subsets of Ω_2,

(Ω_2, F_2) is a measurable space

let P_X be the measure on F_2

$P_X\{0\} = \mu\{HH\} = \frac{1}{4}$

$P_X\{1\} = \mu\{HT, TH\} = \frac{1}{2}$

$P_X\{2\} = \mu\{TT\} = \frac{1}{4}$

Then X is a random variable

ملاحظة اذا كان لدينا space نستطيع ان نعرف اي متغير عشوائي عليه

← measure | random X

يكون على set fun. | يكون على Point fun. مثل Borel measure ايضاً نقطة.

سؤال/

تعريف random variable على فضاء احتمالي (Ω, F, P)

Q/(i) what do we mean by a random variable and by the corresponding probability dist. fun.

what is the expected value of x.

(ii) Given $\Omega = \{TT, FF, FT, TF\}$

x = the number of T in each event

Find $P_x(X = i)$, $i = 0, 1, 2$

and then Calculate $E(x)$.

Def. the distribution fun. (on prob. space)

the dist. fun. of the r.v. X is the fun.
$F = F_X$ from R to $[0,1]$ given by

$F(x) = P\{\omega : X(\omega) \leq x\}$, x real

$\Rightarrow$ $F(b) - F(a) = P\{\omega : a < X(\omega) \leq b\}$, $a, b \in R$, $a < b$

$= P_X(a, b]$

F is a distribution function corresponding to L.s.m. P_X.
to L.s.m. P_X.

(F is increasing, right continuous, $F(x) \to 1$ as $x \to \infty$ and $F(x) \to 0$, as $x \to -\infty$).

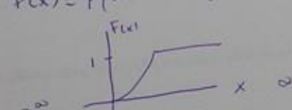
تعريف دالة التوزيع Dist. fun.

$F = F_X$ from $R \to [0,1]$

$F(x) = P\{\omega : X(\omega) \leq x\}$

$F(x) = P(X \leq x)$

$0 \leq F(x) \leq 1$



$0 < p < 1$

Def. if x is a r.v. on $(\Omega, \mathcal{F}, P)$ then

$$E(X) = \int_{\Omega} X \, dP$$ provided by the integral exists.

if g is Borel measurable fun. from $\mathbb{R}$ to $\mathbb{R}$

and $Y = g \circ X$

then $$E(Y) = \int_{\mathbb{R}} g(x) \, dF(x)$$

where. F is the distribution fun. of X

$$\left(= \int_{\mathbb{R}} g \, dP_X \text{ if exists} \right).$$

Remark

A normally distributed r.v. has the usefull property that the dist is completely determined by the mean and the variance.

i.e $E(X) = \mu$, $V(X) = \sigma^2$

and $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Definition and Comments

Discrete متقطع

Def.

let x be a r.v. on $(\Omega, \mathcal{F}, P)$, x is said to be Simple

iff: x can take only finitely many possible value. then

x discrete iff the set of values of x is finite or Countably

infinite if x is discrete and the values of x, $\{x_n\}$, can

be arranged so that $x_n < x_{n+1}$ $\forall n$

then the dist. fun. F is a step fun. with a discontinuity

at each x_n, of magnitude. قفزة

$P_n = P\{X = x_n\}$, F is constant between the x_n and x_{n+1}

and take the upper value at each discontinuity.

(1) if $x_{n-1} < a < x_n \le b < x_{n+1}$ then

(*) $F(b) - F(a) = P\{a < x \le b\} = P_n$

(2) if x is arbitrary determined by the prob. fun. of x are completely determind by the prob. fun. P_X, defined by:

$$P_X(x) = P\{X = x\}, \quad x \in R$$

صريح

B Explicity من منظور رياضي

$$P_X(B) = \sum_{x \in B} P_X(x)$$ (a countable sum P_X is ... = a xcept at the x_n)

(3) if $x_n < c < d < x_{n+1}$ then

$F(d) - F(c) = 0$

متغير عشوائي بسيط   توضيح

simple r. v. X

if	if	if
X arbitrary discrete	$x_n < c < d < x_{n+1}$	$x_{n-1} < a < x_n < b < x_{n+1}$
$P_X(x) = P\{X = x\}$	$F(d) - F(c) = 0$	$F(b) - F(a) = P\{a < x \le b\} =$
		$P_n = P\{X = x_n\}$

Def:

the r.v. X is said to be absolutely cont.'s iff there is a non-negative real valued Borel measurable fun. f on $\mathbb{R}$ such that

$$\int_{-\infty}^{x} f(t)\,dt \quad , x \in \mathbb{R}.$$

we call f the density or the density fun. of X.

since $F(x) \Rightarrow 1$ as $x \to \infty$ we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

if X is absolutely cont.'s with density f it follows that

$$P_X(B) = \int_B f(x)\,dx \quad , \forall B \in \mathcal{B}(\mathbb{R})$$

for the measure μ defined by

$$\mu(B) = \int_B f(x)\,dx \quad , B \in \mathcal{B}(\mathbb{R})$$

isfies $\mu(a,b] = F(b) - F(a)$, $a<b$

us μ is the Lebesgue - stiltjes measure

rresponding to F , ~~there~~ hence
متطابق متشابه

ence $\mu = P_x$

الاستمرارية المطلقة
thus absolutely Continuty of x means

$$P_x \ll L.m.$$

absolutely continuous تعني (<<) هذا الرمز

Any non-negative Borel measurable fun. f on R

with

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

هنا رائد
is the density of some absolutely cont. r.v. x

Def. The r.v. X is said to be cont.s iff its dist. fun. F is cont.s on all of R $\Longleftrightarrow$ X is cont.s iff

iff

$P\{X = x\} = 0 \quad , \forall x$

Def: Random vector: An n-dimensional random vectors on a prob. space (Ω, F, P) is a Borel measurable map from Ω to R^n

two number X and Y are picked at random between 0 and 1 assume that X and Y are indep. and that each is uniformly distributed (i.e. X and Y have densities f_1 and f_2 given by $f_1(x) = f_2(x) = 1$, $0 \le x \le 1$, and 0 o.w.)

let Z be the product XY and let us find the dist. of Z

take $\Omega = \mathbb{R}^2$

$\mathcal{F} = \mathcal{B}(\mathbb{R}^2)$

$X(x,y) = x$

$Y(x,y) = y$

and $f(x,y) = f_1(x) \cdot f_2(y)$

$\therefore \; P\{(x,y) \in B\} = \iint_B f(x,y)\,dx\,dy$

$\Rightarrow P(B) = \iint_B f_1(x) \cdot f_2(y)\,dx\,dy$

$\therefore F(z) = P(Z \le z) = P\{(x,y) : \overbrace{xy \le z}^{B}\}$

$= \iint_{xy \le z} f_1(x) f_2(y)\,dx\,dy$

$\because \; 0 \le x, y \le 1$ i.e x, y between 0, 1 (with prob. 1)

$f(x) = 1 \qquad 0 \le x \le 1$

$f(y) = 1 \qquad 0 \le y \le 1$

$f(x,y) = f(x)\, f(y) = 1 \qquad \begin{matrix} 0 < x < 1 \\ 0 < y < 1 \end{matrix}$

$z = xy$ let $\Rightarrow$ $y = \frac{z}{x} = \frac{z}{g}$

$g = x$ $\qquad x = g$

$\Rightarrow$

$$|J| = \begin{vmatrix} \frac{dy}{dz} & \frac{dy}{dg} \\ \frac{dx}{dz} & \frac{dx}{dg} \end{vmatrix} = \begin{vmatrix} \frac{1}{g} & -\frac{z}{g^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{g}(1) - \left(-\frac{z}{g^2}\right)(0)$$

$$= \frac{1}{g} - 0$$

$$= \frac{1}{g}$$

$J = \frac{1}{g}$

$|J| = \frac{1}{g}$

$$\therefore f(z,g) = \begin{cases} \frac{1}{g} & , 0 < z < 1 \\ & , z < g < 1 \\ 0 & \text{o.w.} \end{cases}$$

$0 < x < 1$
$0 < y < 1$
$0 < xy < 1$
$0 < z < 1$

$x = \frac{z}{y} = g$
$0 < x < 1$
$0 < g < 1$
[illegible]
$y = 0$
$z = y = 0$
$y = 1$
$z = g$

z marginal [illegible] $f(z,g)$ [illegible]

$$\therefore f(z) = \int_z^1 \frac{1}{g}\, dg = \ln g \Big|_z^1 = -\ln z$$

$$f(z) = \begin{cases} -\ln z & , 0 < z < 1 \\ 0 & \text{o.w.} \end{cases}$$

$\therefore\ F(z) = 1$ for $z \geq 1$

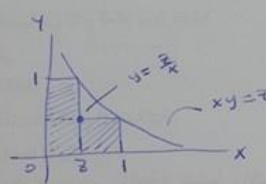
and $F(z) = 0$ for $z \leq 0$

since $f_1(x) \cdot f_2(x)$ is $= 1$ for $0 < x \leq 1$, $0 \leq y \leq 1$

and 0 o.w.

$f(z)$, $(0 < z < 1)$ is the area under the curve xy

~~xyz~~ as fig.1 (where $y = \frac{z}{x}$ if $z < x < 1$



$$F(z) = z + \int_z^1 \frac{z}{x}\, dx\, dx = z - z\ln z \quad , 0 < z < 1$$

$$f(z) = F'(z) = 1 - \left(z\frac{1}{z} + \ln(z) \cdot 1\right)$$

$$= 1 - 1 - \ln z$$

$$= \begin{cases} -\ln z & , 0 < z < 1 \\ 0 & \text{o.w.} \end{cases}$$

Ex. 2

let X, Y and Z be indep. r.v.'s each is normally distributed with $m=0$, $\sigma=1$; i.e X, Y and Z each have the density

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-m)^2}{2\sigma^2}} \quad \text{with } m=0, \text{ and } \sigma^2=1$$

let $w = (x^2+y^2+z^2)$, (take the positive square root so that $w \geq 0$)

Find the dist. fun. of w.

Sol

we take $\Omega = \mathbb{R}^3$,

$$F = \mathcal{B}(\mathbb{R}^3)$$

$$X(x,y,z) = x$$

$$Y(x,y,z) = y$$

$$Z(x,y,z) = z$$

and

$$P(B) = \iiint_B f(x,y,z)\,dx\,dy\,dz$$

where $f(x,y,z) = g(x)\,g(y)\,g(z)$

$$f(x,y,z) = g(x)\cdot g(y)\cdot g(z)$$

$$= (2\pi)^{-\frac{3}{2}} \exp\left\{-\tfrac{1}{2}(x^2+y^2+z^2)\right\}$$

then

$$F(\omega) = P\{W \le \omega\} = P\{x^2+y^2+z^2 \le \omega^2\} \quad \text{if } \omega \ge 0$$

$$\therefore F(\omega) = \iint\limits_{x^2+y^2+z^2 \le \omega^2} (2\pi)^{-\frac{3}{2}} \exp\left\{-\tfrac{1}{2}(x^2+y^2+z^2)\right\} dx\,dy\,dz$$

بالانتقال الى r نستخدم الاحداثيات الكروية لايجاد ناتج التكامل

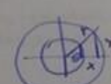
in spherical coordinates.

$$F(\omega) = \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^{\omega} (2\pi)^{-3/2} \exp\left\{-\tfrac{1}{2}r^2\right\} \cdot r^2 \sin\phi \, dr$$

$$= (2\pi)^{-3/2}(2\pi)(2)\int_0^{\omega} r^2 e^{-\frac{1}{2}r^2} dr$$

$\therefore$ W is absolutely continuous with density

$$f(\omega) = \begin{cases} \dfrac{2}{\sqrt{2\pi}}\, \omega^2 e^{-\frac{1}{2}\omega^2}, & \omega \ge 0 \\ 0 & \text{o.w.} \end{cases}$$



$r^2 = x^2+y^2$

$r^2 = x^2+y^2+z^2$

Theorem

let $x_1, x_2, \dots, x_n$ be indep. r.v.s on (Ω, F, P)

if all x_i are non-negative or if Ex_i is finite, $<\infty$

for all i, then $E(x_1 x_2 \dots x_n)$ exists and equals

$E(x_1) \cdot E(x_2) \dots E(x_n)$

proof ① if all $x_i \geq 0$ then non-negative

$$E(x_1, \dots, x_n) = \int_R x_1 \dots x_n \, dP_x(x_1, \dots, x_n) \qquad x = x_1, \dots, x_n$$

since P_x is the product of P_{x_i} (using Fubini's theorem)

we get

$$E(x_1, \dots, x_n) = \int_R x_1 \, dP_{x_1}(x_1) \cdot \int_R x_2 \, dP_{x_2}(x_2) \dots \int_R x_n \, dP_{x_n}(x_n)$$

$$E(x_1, \dots, x_n) = E(x_1) \cdot E(x_2) \dots E(x_n)$$

② if all x_i finite then.

$$E(|x_1, \dots, x_n|) = \prod_{i=1}^{n} E(x_i) < \infty$$

Thus Fubinis theorem may be applied just as in the first part of the proof.

Theorem Cauchy-Schwartz inequality

let X and Y have finite second moments then

$$[E(XY)]^2 = |E(XY)|^2 \le EX^2 \cdot EY^2$$

proof.

Corollary: $|\rho_{XY}| \le 1$, this equality holds iff:

$P[Y = cX] = 1$ for some constant c.

theorem ~~chebyshev's~~ Cauchy, -sshwartz inequality

let x and y have finite second moments then.

$$[E(xy)]^2 = |E(xy)|^2 \leq Ex^2 . Ey^2$$

proof :

corollary. $|\rho_{xy}| \leq 1$, this equality holds iff:
$P\{Y = cx\} = 1$ for some constant c.

conditional expectation.

Theorem. let X and Y be a two dimensional r.v. each then $u_1(x)$, $u_2(y)$ be a fun. of r.v. then

(i) $E[u_1(Y) + u_2(Y) | X = x] = E[u_1(Y) / X = x] + E[u_2(Y) | X = x]$

(ii) $E\{u_1(X) . u_2(Y) | X = x\} = u_1(x) . E\{u_2(Y) | X = x\}$

proof.

Note if x and y independent then.

(i) $f(x|y) = f(x)$

(ii) $p(a<x<b \mid Y=y) = p(a<x<b)$

(iii) $E(X \mid Y=y) = E(X)$

Theorem let x and Y be a two dimensional r.v. then if $E[u(Y)]$ exists we have

$$E\{u(Y)] = E\{E\{u(Y) \mid X=x\}\}$$

Theorem.

$$Var(Y) = E\left[Var[Y \mid X = x]\right] + Var\left[E(Y \mid X = x)\right]$$

proof

proof

$$E(y) = E\{E[y \mid X = x]\}$$

$$E(x) = E\{E\{x \mid Y = y\}\}$$

example.

let $f(x,y) = x + y \quad , 0 < x < 1 \quad , 0 < y < 1$

Find ① $E\{X \mid Y = y\}$

② $E\{X \mid Y = \frac{1}{2}\}$

③ $E\{X + 2X^2 \mid Y = \frac{1}{2}\}$

④ $E\{X^1 Y^2 \mid Y = \frac{1}{2}\}$

⑤ $Var[X \mid Y = \frac{1}{2}]$

example let $X \sim Ber(p)$. suppose that.

$E[Y \mid X = 0] = 1$ and $E[Y \mid X = 1] = 2$

what is EY.

example/ let $f(x) = \begin{cases} \frac{x}{3} & , x = 1, 2 \\ 0 & o.w. \end{cases}$

and let $Y \mid X = x \sim b\{x, \frac{1}{2}\}$

Find ① EY

② $f(x,y)$.

how that. $E[Var[Y|X=x]] = Var(Y)$ iff.
$E[Y|X=x]$ is independent of x.

* show that $X+Y$ and $X-Y$ are uncorrelated iff
$\sigma_x^2 = \sigma_y^2$

* if $\sigma_x^2 = \sigma_y^2$ for any r.v.s X and Y then show that

$$\rho_{x,x+y} = \cancel{\frac{Cov(X, X+Y)}{\sqrt{V(X)\, V(X+Y)}}} = \sqrt{\tfrac{1}{2}(1+\rho_{x,y})}$$

* show that for any indep. r.v.'s X and Y then
$Var(Y) = \mu_y^2 \sigma_x^2 + \mu_x^2 \sigma_y^2 + \sigma_x^2 \sigma_y^2$.

* if X and Y r.v. and independent and if $U_1(x)$ and $U_2(y)$
are fun. for. the two r.v. then.
$E[U_1(X), U_2(Y)] = E[U_1(X)] \cdot E[U_2(Y)]$

* let the joint p.d.f. of X and Y be given by
$f(x,y) = 1 - a(1-2x)(1-2y)$ $\quad 0<x<1$, $0<y<1$
prove or disprove that X and Y are indep. iff.
they are uncorrelated

...bini's theorem

...sume the hypothesis. of the product measure theorem

which says that: let (Ω, F_1, μ_1) be a measure space
with μ_1 σ-finite on F_1
and let Ω_2 be a set with σ-field ~~assume~~ that
~~for each $\omega \in \Omega_1$~~
assume that for each $\omega_1 \in \Omega_1$.

[illegible]

we are given a measure $\mu(\omega_1, \cdot)$ on F_2

assume that $\mu(\omega_1, B)$, besides being a measure
in B for each fixed $\omega_1 \in \Omega$, is Borel measurable
in ω_1 for each fixed $B \in F_2$.

Assume that $\mu(\omega_1, \cdot)$ are uniformly σ-finite
(i.e. Ω_2 can be written as $\bigcup_{n=1}^{\infty} B_n$).

where for some positive (finite) constant K_n
we have $\mu(\omega_1, B_n) \leq K_n$, $\forall \omega_1 \in \Omega_1$

[The case in which $\mu(\omega_1, \cdot)$ are uniformly bounded
i.e $\mu(\omega_1, \Omega_2) \leq K < \infty$]

$\rightarrow$ Then there is a unique measure μ on $F = F_1 \times F_2$

such that

$$\mu(A \times B) = \int_A \mu(\omega_1, B)\, d\omega_1 \quad \forall A \in F_1, B \in F_2$$

انتهت نظرية حاصل الضرب
لنظرية فوبيني.

let $f : (\Omega, \mathcal{F}) \longrightarrow (\bar{R}, B(\bar{R}))$

if f non- negative.

then

$$= \int_{\Omega_2} f(\omega_1, \omega_2)\, \mu(\omega_1, d\omega_2) \text{ exists and defines}$$

a Borel measurable to ω_1

also

$$\int_{\Omega} f\, d\mu = \int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2)\, \mu \omega_1, d\omega_2 \right) \mu(d\omega_1)$$

proof

$$E[g_1(x)] \le E[g_2(x)] \quad , \text{if} \quad g_1(x) \le g_2(x).$$

proof $\quad |E(g(x))| \le E[|g(x)|]$