P-264-265

example

The number of students that have enrolled in a prob. course is a $\boxed{\text{poisson}}$ random variable with mean $\boxed{80}$. The professor in charge of the course has decided that if the number enrolling is $\boxed{100}$ or $\boxed{\text{more}}$ he will teach the course in two seperate sections. otherwise he will teach it in one section. What is the prob. that the professor will teach the course in one section.

Solution

The required prob. is.

$$p\{x < 100\} = \sum_{k=0}^{99} \frac{e^{-80}(80)^k}{k!}$$

if we assume that the poisson random variable with mean $\boxed{80}$ is the sum of $\boxed{80}$ independent poisson random variables. each with mean $\boxed{1}$, then by Centeral limit theorem. we have.

$$p(x < 100) = p\left(\frac{x - 80}{\sqrt{80}} < \frac{100-80}{\sqrt{80}}\right)$$

$$= p\left(z < \frac{20}{8.944}\right)$$

$$= \phi(2.396)$$

$$= \boxed{0.9916}$$

problem

① show that if $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{d} X$

② let $\{X_n\}$ be asequence of independent r.v.s. such that X_n has binomial dist. with parameters $\boxed{n}$) and $\boxed{P}$

if $Z_n = \dfrac{X_n - E(X_n)}{\sqrt{Var(X_n)}}$

then use the Centeral limit theorem. to show that

Z_n has $N(0,1)$ as $n \longrightarrow \infty$

(3) Repeat problem ② if X_n has apoisson dist. with paramete

(4) let x be the number of successes in 5000 Bernoulli trials write probability of successes of 0.7 on agiven trials Use the Centeral limit theorem to estimate

① $p(x < 4950)$

② $p(3920 < x < 4240)$

1. an unbiased die is thrown 480 times find the lower bound for the prob. of getting 60 to 100 sixes. (By using the chebyshev inequality)

Sol.

let x be the number of sixes obtained then x has binomial with parameter

$n = 480$, $p = \frac{1}{6}$

Thus

$$E(x) = np = (480)\left(\frac{1}{6}\right) = 80$$

$$V(x) = npq = (480)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = 66.666$$

by using the chebyshev's inequality

$$P\{|x - E(x)| < \epsilon\} \geq 1 - \frac{Var(x)}{\epsilon^2}$$

we get.

$$P\{-\epsilon < x - 80 < \epsilon\} \geq 1 - \frac{66.666}{\epsilon^2}$$

or

$$P\{80 - \epsilon < x < 80 + \epsilon\} \geq 1 - \frac{66.666}{\epsilon^2}$$

$80 - \epsilon < x < 80 + \epsilon$

$60 < x < 100$

$80 - \epsilon = 60$

$80 - 60 = \epsilon$

$20 = \epsilon$

$80 + \epsilon =$

$\epsilon = 10$

$\epsilon = 2$

وهذا [illegible]

$\epsilon = 20$

Now, compare the last inequality with

$P(60 < x < 100)$ we have

$\epsilon = 20.$

Therefore

$$P\{60 < x < 100\} \geq 1 - \frac{66.666}{400}$$

$$= 1 - 0.166$$

$$= 0.834$$

xample p. 175. prob. Dr. Kubais 1993

Two unbiased dice are thrown if X is the sum of the numbers showing up prove that.

$$P\{|X-7| \geq 3\} \leq \frac{35}{54}$$

p. 176.

example Suppose that it is know that the number of items produced in a factory during a week is a r.v. with amean 600.

(1) what can be said about the prob. that this weeks production will exceed 720

(2) if the variance of a week's production is known to equal 20 then what can be said about the prob. that this week's production will be between 590 and 610.

5)

__example__ Two unbiased dice are thrown if x is the sum of the numbers showing up, prove that.

$$p\{|x-7| \geq 3\} \leq \frac{35}{54}$$

__sol__ By Chebyshev's inequality for $\epsilon > 0$, we have

$$p\{|x-m| \geq \epsilon\} \leq \frac{var(x)}{\epsilon^2}$$

we have

$$m = E(x) = \Sigma x\, p(x)$$

$$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right)$$

$$+ 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right)$$

$$+ 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$$

$$= \boxed{7}$$

and $Ex^2 = \Sigma x^2 p(x)$

$$= \frac{1}{36}\big(1(2)^2 + 2(3)^2 + 3(4)^2 + 4(5)^2 + 5(6^2) + 6(7)^2$$

$$+ 5(8)^2 + 4(9)^2 + 3(10)^2 + 2(11)^2 + (1)(12)^2$$

$$Ex^2 = \frac{329}{6}$$

Thus $var(x) = Ex^2 - [E(x)]^2$

$$= \frac{329}{6} - (7)^2 =$$

$$= \frac{35}{6}$$

Then we have $m = 7$

$$\sigma^2 = \frac{35}{6}$$

$$\epsilon = 3$$

There fore.

There fore.

$$P\{|X-7| > 3\} \le \frac{35/6}{9} = \frac{35}{54}$$

But the actual prob. is.

$$P\{|X-7| \geq 3\} = P\{X-7 \geq 3 \text{ or } X-7 \le -3\}$$

$$= P\{X \geq 10 \text{ or } X \le 4\}$$

$$= P\{X \geq 10\} + P\{X \le 4\}$$

$$= \frac{3}{36} + \frac{2}{36} + \frac{1}{36} + \frac{1}{36} + \frac{2}{36} + \frac{3}{36}$$

$$= \frac{12}{36}$$

$$= \frac{1}{3}$$

Segence of random variables

let us recall the definition of convergence of a sequence of real numbers $X_1, X_2, \dots$ to a real numbers X, the sequence $[X_n]$ is said to be convergent to a limit X, if for any $\epsilon > 0$, chosen as small as possible, there is a positive integer m, such that for $n > m$ we have

$$\{X_n - X\} < \epsilon \text{ for all } n > m$$

Note let $X_1, X_2, \dots$ be a sequence of r.v.s define on the same prob. space S. we shall assume that any finite collection $(X_1, X_2, \dots, X_n)$. $(n > 1)$ for n random variables are jointly distributed. that is they have a joint dist. $F_n(x_1, \dots, x_n)$ on R^n (n-n-Euclidean space) which is defined as,

$$F_n(x_1, \dots, x_n) = P\{X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n\}$$

where $x_1, x_2, \dots, x_n$ are real numbers, The dist. Fun. F_n satisfies the following properties

F_n satisfies the following properties.

① $0 \leq F_n(x_1, x_2, \dots, x_n) \leq 1$

② $\lim\limits_{x_i \to -\infty} F_n(x_1, \dots, x_i, \dots, x_n) = 0$, for all $i = 1, 2, \dots, n$

③ $\lim\limits_{\text{all } x_i \to \infty} F_n(x_1, \dots, x_i, \dots, x_n) = 1$, $i = 1, 2, \dots, n$

④ $F_n(x_1, \dots, x_n)$ is right continuous in every ~~convergence~~ variable.

Lect 13

→ in probability theory there are 4 kinds of conver
of the sequence $[X_n]$ of r.v.'s which are.

① convergence in prob. (or stochastic convergence

② convergence almost surely (or strongly)

③ convergence in Mean

④ convergence in Distribution

what is the kind of convergence in prob. theory.
for the sequence $[X_n]$ of the r.v.s.
(talk about each one).

1- Convergence in probability
(or stochastic convergence).

a sequence of r.v.'s $\{X_n\}, \{n \geq 1\}$ is said to be convergent in probability to a r.v. X. (or weakly convergent). written as:

$$X_n \xrightarrow{P} X, \text{ if for every } \epsilon > 0,$$

$$\lim_{n\to\infty} P\{|X_n - X| \geq \epsilon\} = 0$$

The equation means that the prob. of the event $\{|X_n - X| \geq \epsilon\}$ tend to zero as $n \to \infty$

also it can be written as

$$P\{|X_n - X| < \epsilon\} \to 1 \quad \text{as } n \to \infty$$

in particular, the sequence $\{X_n\}$ converges stochastically to a constant a, $(a \geq 0)$;

written as $\quad X_n \xrightarrow{P} a$

if for every $\epsilon > 0$

$$\lim_{n\to\infty} P\{|X_n - a| \geq \epsilon\} = 0$$

if $X_n \xrightarrow{P} a$ and $Y_n \xrightarrow{P} b$ as $n \to \infty$
then we have

(i) $X_n \mp Y_n \to a \mp b$ as $n \to \infty$

(ii) $X_n Y_n \to a . b$ as $n \to \infty$

(iii) $\dfrac{X_n}{Y_n} \to \dfrac{a}{b}$ as $n \to \infty$

قسي
P.157

2- convergence almost surely

a sequence of r.v.'s $\{X_n\}$ is said to be convergent almost surely (or strongly) to a r.v.'s X, written as

$$X_n \xrightarrow{a.s} X \quad \text{if}$$

$\xrightarrow{a.s}$ almost strongly or almost surely.

$$P\{\lim_{n \to \infty} X_n = X\} = 1$$

in this case we write $\lim_{n \to \infty} X_n = X$

with prob. one (simply w.p.1)

in other words, for given $\epsilon > 0$

if for almost all $\omega \in S$ (sample space) on which the r.v.'s $\{X_n\}$ and X are defined we have.

$$P\left[\bigcup_{n=N}^{\infty} \{|X_n(\omega) - X(\omega)| > \epsilon\}\right] \to 0$$

as $N \to \infty$ that is, the set of all $\omega \in S$ for which $X_n(\omega)$ does not converge to $X(\omega)$ as $n \to \infty$ in an event and has prob. zero, then the sequence X_n a.s. X.

$$X_n \xrightarrow{a.s} X$$

3- Convergence in Mean

a sequence of r.v.'s $\{X_n\}$ is said to be Convergent in r-th mean to a r.v. X if

$$E X_n^r < \infty \quad , \quad E X < \infty$$

and

$$\lim \left[E | X_n - X |^r \right] = 0 \quad , \quad r > 0$$

we use the notation

rth-mean

$$X_n \xrightarrow{r.m} X$$

The case $r=2$ is known as convergence in a quadratic mean. or mean square convergence. in this case we write

$$X_n \xrightarrow{m.s.} X$$

p. 258

Theorem

if a sequence of r.v.'s $\{X_n\}$, converges in mean square to X. then it also converges in probability to X.

mean $X_n \xrightarrow{P} X$

prob.

proof

Recall the chebyshev's inequality we have

for any $\epsilon > 0$

$$P\{|X_n - X| \geq \epsilon\} \leq \frac{E[|X_n - X|^2]}{\epsilon^2}$$

and for fixed $\epsilon > 0$, we have

$$\lim_{n\to\infty} P\{|X_n - X| \geq \epsilon\} \leq \frac{1}{\epsilon^2} \lim E\{|X_n - X|^2\}$$

therefore

$$X_n \xrightarrow{P} X$$

p258

4- <u>Convergence in Distribution</u>

The sequence $\{X_n\}$ of r.v.'s is said to be convergent in dist. (or in law) to the r.v. X. if the dist. fun. $F_n(x)$ of X_n ~~converges~~ to the dist. fun. $F(x)$ of X at every continuity point x of F.

it is written as

$$X_n \xrightarrow{d} X$$

Theorem

if $X_n \xrightarrow{P} X$ the the dist. fun. $F_n(x)$ of X_n tends to the dist. fun. $F(x)$ of X at every continuity point of $F(\ )$.

(For the proof see Srinivasan and Mehata, P. 234.)

Defintion

let $X_1, X_2, \dots, X_n, \dots$ be asequence of r.v's with dist. fun. $F_n(x)$, $(n=1,2,\dots)$ we said that the sequence of dist. funistions $[F_n(x)]$ is convergent if there exists adist. fun. $F(x)$ such that at every Continuity point of $F(x)$ the relation,

$$\lim_{n\to\infty} F_n(x) = F(x)$$

is hold.

To avoid the difficulty in finding the limit dist. fun specially that this limit is not defined at the discontinuity points, we use another approach which will be clarified by the following theorem (also for the proof of this theorem, see Srinivasan and Mehata pp. 238-242)

Theorem

let $\phi(t)$ be the characteristic fun. of X_n.
if $X_n \xrightarrow{d} X$ then $\phi_n(t) \to \phi(t)$, where
$\phi(t)$ is the c.d.f. of X. if $\phi_n(t) \to \phi(t)$
and the limit fun. is continuous at $t=0$,
then

$$X_n \xrightarrow{d.} X$$

The law of the large number

in many experiments empirical data appear to obey acertain general law.. that is, if an experiment is repeated $\boxed{n}$ times and if an event $\boxed{E}$ occure with probability $\boxed{P}$ then with prob. one the proprotion of successe of $\boxed{E}$ approach $\boxed{P}$. This results is known as the strong law of large numbers.

Bernoulli's law of large Numbers.

An $\boxed{n}$ independent Bernoulli trials with probability of success $\boxed{p}$. is performed.

if $\boxed{X}$ is the number of successes in these $\boxed{n}$ trials then $\boxed{E(X) = np}$ and $\boxed{Var(X) = npq}$. (Since X has a binomial dist. with parameter $\boxed{n}$ and $\boxed{p}$,

The variable $\boxed{\frac{X}{n}}$ represents the proportion of successes, we have

$$E\left(\frac{X}{n}\right) = \frac{E(X)}{n} = \frac{np}{n} = \boxed{p}$$

and $$Var\left(\frac{X}{n}\right) = \frac{1}{n^2} Var(X) = \frac{1}{n^2} npq = \boxed{\frac{pq}{n}}$$

Then for given $\epsilon > 0$

$$P\left\{\left|\frac{X}{n} - p\right| \geq \epsilon\right\} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \quad \ldots (\#\#)$$

From this equation (##) it follows that:

$$\frac{X}{n} \xrightarrow{P} p \quad \text{as} \quad n \longrightarrow \infty$$

The weak law of large number :

This theorem deals with the fact that under some conditions the average of asequence of random variables converges to the expected average.

Theorem

let $x_1, x_2, \ldots, x_n$ be arandom Sample of size n drawn from apopulation with a finite mean μ and let $\boxed{\bar{X}_n}$ be the Sample mean of the random Sample then for any $\epsilon > 0$. we have

$$P[|\bar{X} - \mu| > \epsilon] \to 0 \quad \text{as} \quad n \longrightarrow \infty$$

P 260-261

<u>Theorem</u> The weak law of the large numbers

Let $X_1, X_2, \dots, X_n$ be a sequence of independent and identically distributed random variables. each having a finit mean

$\mu = E(X_i)$, $i = 1, 2, \dots$

Then for any $\epsilon > 0$ and $var(X_i) = \sigma^2 < \infty$

we have

$$P\left\{ \left| \frac{X_1 + X_2 + \dots + X_n}{n} - \mu \right| \geq \epsilon \right\} \longrightarrow 0 \quad \text{as } n \to \infty$$

<u>proof</u>:

if we write

$$\bar{X}_n = \frac{X_1 + X_2 + \dots + X_n}{n}$$

then

$$\{ | \bar{X}_n - \mu | \geq \epsilon \} \to 0 \text{ as } n \to \infty$$

this means

$$\bar{X}_n \xrightarrow{P} \mu \quad \text{as } n \to \infty$$

where $\mu = E(\bar{X}_n)$

By using Chebyshev's inequality it can be easily proved.

if X_i , $i = 1, 2, \dots$ are independent r.v.'s with a finite mean $E(X_i) = \mu_i$ for $i = 1, 2, \dots$ then

$$\bar{X}_n \xrightarrow{P} \bar{\mu}_n \quad \text{as } n \to \infty$$

where $\bar{\mu}_n = E(\bar{X}_n) = \frac{\sum \mu_i}{n}$

provided that

$$\lim Var(\bar{X}_n) = 0$$

The strong law of large numbers

let $X_1, X_2, \ldots, X_n, \ldots$ be a independent and identically dist. with a finite mean $\mu = E(X_i)$ and finite fourth centeral moment $\mu_4 = E(X_i - \mu)^4$, for $i = 1, 2, \ldots$

Then

$$\bar{X}_n \xrightarrow{a.s.} \mu$$

That is

$$P\left\{\lim_{n\to\infty} \bar{X}_n = \mu\right\} = 1$$

Weak law of Large numbers

let $x_1, x_2, \ldots$ be indep. r.v.s (not necessarily with the same dist.)
each with finite mean and variance.

Assume the variance to be uniformly bounded by $u < \infty$ نهاية

let $S_n = x_1 + x_2 + \cdots + x_n$ then

$\frac{S_n - E(S_n)}{n}$ converge in prob. to 0 صفر

i.e given $\epsilon > 0$ then

$$P\left\{ \left| \frac{S_n - E(S_n)}{n} \right| \geq \epsilon \right\} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$P(X \geq \epsilon] \leq \frac{EX^r}{\epsilon^r} \quad \rightarrow \text{chebyshev inequ.}$$

proof By chebyshev's inequality

$$P\left\{ \left| \frac{S_n - E(S_n)}{n} \right| \geq \epsilon \right\} \leq \frac{1}{\epsilon^2} E\left[\left(\frac{S_n - ES_n}{n} \right)^2 \right]$$

$$\leq \frac{1}{\epsilon^2 n^2} Var(S_n)$$

$$\leq \frac{1}{\epsilon^2 n^2} \sum_{k=1}^{n} Var(X_k)$$

[illegible]

$$\leq \frac{\mu^2}{\epsilon^2 n} \longrightarrow 0 \quad \text{as } n \to \infty$$

There are two special cases of particular interest

① if $EX_i = m \ \forall i$

$ES_n = \frac{1}{n}(EX_1 + EX_2 + \cdots + EX_n) = \frac{1}{n}$

then $\frac{S_n - E(S_n)}{n} = \frac{S_n - nm}{n} = \frac{S_n}{n} - \cancel{m}$

[illegible] $m = \bar{X}$ $= EX$

hence.

$\frac{S_n}{n} \rightarrow m$ in probability

$n \to \infty$ mean $\bar{x} = E\bar{x}$

Thus for large n, the arithimatic average of n indep. r.v.'s each with finite expectation m (and ~~the~~ with the variance uniformly bounded)

is quite likely to be very close to m.

(2) if $X_1, \dots,$ ... are indep. and for each i,

$P\{X_i = 1\} = p$, $P\{X_i = 0\} = q = 1 - p$

{They we have an infinite sequence of Bernoulli trials)

Then $X_1 + X_2 + \dots + X_n$ is the number of successes in trials

hence $\frac{S_n}{n}$ is the relative frequency of successes

since $E(X_i) = P$,

we have $\frac{S_n}{n} \longrightarrow P$ in probability

thus : for Large n, the realtive frequency of successes is quite likely to be very close to P.

weak law of large number

Two special cases

$$\frac{S_n - E(S_n)}{n}$$

$x \sim Ber(1, p)$
$Ex = p$
$v(x) = pq$

عادلة برنولي

نجاح $p(X_i = 1) = p$

فشل $p(X_i = 0) = q = 1 - p$

$E(x_i) = p$

$\frac{S_n}{n} \to p$

$\frac{S_n}{n} \to p$ in prob.

for large number.

$\frac{S_n}{n}$

relative frequency successes is quite likely to be very close to p

$E(x_i) = m$

$\frac{S_n}{n} - m$

$\frac{S_n}{n} \to m$ in prob.

in a large number

arithmatic average of finite expectation is quite likely to be very closed to m

Kolumogrove strong law of large number

let $X_1, X_2, \ldots$ be indep. r.v.s each with finite mean and variance and let $\{b_n\}$ be an increasing seq. of positive real numbers with $b_n \to \infty$

if $$\sum_{n=1}^{\infty} \frac{Var(X_n)}{b_n^2} < \infty$$

الموضوع الأساسي
as $n \to \infty$
$n = b_n$

then (with $S_n = X_1 + X_2 + \cdots + X_n$)

$$\frac{S_n - E(S_n)}{b_n} \to 0 \quad \text{a.e}$$

خصوصا
in particulare

if the X_n are independent r.v.'s each with finite mean m, finite variance σ^2, then

$$\frac{S_n}{n} \to m \quad \text{a.e}$$ (check take $b_n = n$)

Liapounov's Condition

Assume that

$$\frac{1}{C_n^{2+\delta}} \sum_{k=1}^{n} E\left[\,|X_k - m_k|^{2+\delta}\,\right] \longrightarrow 0$$

for som $\delta > 0$ then

$$E\left[\,|X_k - m_k|^{2+\delta}\right] = \int_{-\infty}^{\infty} |x - m_k|^{\delta}\, |x - m_k|^{2}\, dF_X(x)$$

$$\geq \int_{\{x : |x - m_k| \geq \epsilon C_n\}} (x - m_k)^2\, dF_X(x)$$