5-2 Continuous Probability Distributions

5-2-1 Uniform Distribution: $X \sim U(a,b)$

A r.v. X is said to have uniform distribution in the interval $[a,b)$ if its p.d.f is given by:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & o.w \end{cases}$$

a, b real constants
$a < b$

The distribution function F is given by:

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x < b \\ 1 & b \leq x \end{cases}$$

prove it?

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du = \int_{a}^{x} \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_{a}^{x} dx$$

$$= \frac{1}{b-a}\Big[x\Big]_{a}^{x} = \frac{x-a}{b-a}$$

examples : The following p.d.f's are some examples of uniform distribution :

1. $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & o.w \end{cases}$ $\quad \frac{1}{b-a} = \frac{1}{1-0} = 1$

2. $f(x) = \begin{cases} \frac{1}{5} & 2 < x < 7 \\ 0 & o.w \end{cases}$ $\quad \frac{1}{b-a} = \frac{1}{7-2} = \frac{1}{5}$

3. $f(x) = \begin{cases} \frac{1}{4} & -1 < x < 3 \\ 0 & o.w \end{cases}$ $\quad \frac{1}{b-a} = \frac{1}{3+1} = \frac{1}{4}$

$\boxed{\text{ex. } 5}$ let $x \sim u(5,10)$. 1. Write the p.d.f of x ?
2. What is the c.d.f of x ?

sol. $f(x) = \frac{1}{b-a} \qquad a < x < b$

$\boxed{1.}$ $\Longrightarrow x \sim u(5,10) \Longrightarrow a = 5 \;,\; b = 10 \Longrightarrow f(x) = \frac{1}{10-5}$

$\Longrightarrow f(x) = \begin{cases} \frac{1}{5} & 5 < x < 10 \\ 0 & o.w \end{cases}$

$\boxed{2.}$ $F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & b \le x \end{cases} \Longrightarrow F(x) = \begin{cases} 0 & x < 5 \\ \dfrac{x-5}{5} & 5 \le x < 10 \\ 1 & 10 \le x \end{cases}$

OR $F(x) = p(x \le x) = \int_5^x \frac{1}{5}\, dx = \frac{1}{5}\int_5^x dx = \frac{1}{5}\left[u\right]_5^x$

$F(x) = \dfrac{x-5}{5}$

$\boxed{\text{ex. 6}}$

Buses arrive at a specified stop at 15 minutes intervals starting at 7:00 a.m. That is they arrive at 7:00, 7:15, 7:30, 7:45, and so on, if a passanger arrives at the stop at a time that is uniformly distributed between 7:00 and 7:30. Find the probability that he waits:

a. less than 5 minutes for a bus.

b. more than 10 minutes for a bus.

Sol. $X \sim U(0, 30) \Longrightarrow f(x) = \begin{cases} \dfrac{1}{30} & 0 < x < 30 \\ 0 & o.w. \end{cases}$

a. p(he waits less than 5 minutes for a bus)

$= p(10 < x < 15) + p(25 < x < 30)$

$= \int_{10}^{15} \frac{1}{30}\, dx + \int_{25}^{30} \frac{1}{30}\, dx = \frac{1}{30}\int_{10}^{15} dx + \frac{1}{30}\int_{25}^{30} dx$

$$= \frac{1}{30} \times \Big]_{10}^{15} + \frac{1}{30} \times \Big]_{25}^{30} = \frac{1}{30}(15-10) + \frac{1}{30}(30-25)$$

$$= \frac{1}{30}(5) + \frac{1}{30}(5) = \frac{10}{30} = \boxed{\frac{1}{3}}$$

$$= P(10 < X < 15) + P(25 < X < 30)$$

$$\therefore \int_{10}^{15} f(x)\,dx + \int_{25}^{30} f(x)\,dx$$

$$\text{or} \quad F(15) - F(10) + F(30) - F(25)$$

$$\text{where } F(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$$

$a = 0,\ b = 30$ then

$$F(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{x}{30} & 0 < x < 30 \\ & x \ge 30 \end{cases}$$

$$= F(15) - F(10) + F(30) - F(25)$$

$$= \frac{15}{30} - \frac{10}{30} + \frac{30}{30} - \frac{25}{30} = \frac{5}{30} + \frac{5}{30} = \boxed{\frac{1}{3}}$$

b. p(he waits more than 10 minutes for a bus) =

$$= p(0 < x < 5) + p(15 < x < 20)$$

$$= \int_0^5 \frac{1}{30}\,dx + \int_{15}^{20} \frac{1}{30}\,dx$$

$$= \frac{1}{30}\left[x\right]_0^5 + \frac{1}{30}\left[x\right]_{15}^{20}$$

$$= \frac{1}{30}(5-0) + \frac{1}{30}(20-15)$$

$$= \frac{5}{30} + \frac{5}{30} = \frac{10}{30} = \boxed{\frac{1}{3}}$$

$p(0 < x < 5) + p(15 < x < 20)$

or By using c.d.f.

$$= F(5) - F(0) + F(20) - F(15)$$

$$= \frac{5}{30} - (0) + \frac{20}{30} - \frac{15}{30}$$

$$= \frac{5}{30} + \frac{5}{30} = \frac{10}{30} = \boxed{\frac{1}{3}}$$

5.2-2 The Normal distribution $X \sim N(\mu, \sigma^2)$

A r.v. X is said to have anormal distribution with parameters μ and σ^2 if it's p.d.f is given:

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} & , -\infty < x < \infty \\ & , -\infty < \mu < \infty \\ & , \sigma^2 \geq 0 \\ 0 & , o.w \end{cases}$$

, μ and σ are real constant.

The values μ represents the mean of X and the value σ^2 represent the variance of X.

الشكل له بيل جرس.

سيتم التركيز فقط على كيفية حساب المساحة تحت المنحني، اما تكامل الدالة خلال تساوي واحد فهو غير مطلوب اي

$$\int_{-\infty}^{\infty} f(x)\, dx.$$

وكذلك فاننا لا نحسب الاحتمال او المساحة تحت المنحني باستخدام p.d.f وكذلك لا نجد c.d.f من استخدام p.d.f وذلك بسبب صعوبة ذلك. هذه بعيدا.

$$F(a) = \int_{-\infty}^{a} f(x)\, dx = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$$F(x) = P(X < x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

ولصعوبة استخدام p.d.f او c.d.f فاننا سنستخدم لحساب التوزيع الطبيعي القياسي Standard Normal distribution

Notes:

1. If $X \sim N(\mu, \sigma^2)$, then $Z = \left(\frac{X-\mu}{\sigma}\right)$ is a standard Normal distribution with mean zero and variance 1.

   This mean $\overset{X \sim N(0,1)}{Z} \sim N(0,1)$.

2. The p.d.f of the standard Normal dist. is:

$$f(z) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} & , z > 0 \\ 0 & , o.w \end{cases} \qquad z \sim N(0,1)$$

3. $P(a < Z < b) = \phi(b) - \phi(a)$ or $= N(b) - N(a)$ where $\phi(b)$ and ϕ are calculate from table.

4. if $X \sim N(\mu, \sigma^2)$ and a & b are constant then

$$P(a < X < b) = \left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$$

$$= \left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= \underbrace{N\left(\frac{b-\mu}{\sigma}\right)}_{\text{من الجدول}} - \underbrace{N\left(\frac{a-\mu}{\sigma}\right)}_{\text{من الجدول}}$$

* لا يمكن إيجاد الاحتمال بالأنتج $P(a < X < b)$ أي $f(x)dx$ or $F(b) - F(a)$ لأن دالة التوزيع هي $F(x)$ وذلك طرح μ وقسمة على σ.

(8)

(5) $p(Z < a) = \phi(a) = N(a)$ → from table.

(6) $p(X < a) = p\left(\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right)$

$= p\left(Z < \frac{a-\mu}{\sigma}\right) = N\left(\frac{a-\mu}{\sigma}\right)$ → from table.

$= \phi\left(\frac{a-\mu}{\sigma}\right)$

(7) $p(X > a) = 1 - p(X < a)$ متوسط منية أكبر من الجدول الاحتمالي لذلك نستخدم سالب.

$= 1 - N\left(\frac{a-\mu}{\sigma}\right)$

(8) $p(Z < -a) = p(Z > a) =$

$= 1 - p(Z < a)$

$= 1 - N(a)$

$p(Z > -a) = 1 - p(Z < -a)$

$1 - \phi(-a) = 1 - (1 - \phi(a)) = 1 - 1 + \phi(a) = \phi(a)$

$p(Z < -a) = \phi(-a)$

$= 1 - \phi(a)$.

$p(Z < a) = \phi(a)$

$p(Z < -a) = \phi(-a)$

$\phi(-a) = 1 - \phi(a)$

(9) $N(-a) = 1 - N(a)$.

ملاحظة

$p(X \geq z)$

$p(X > z)$

هنا المساواة غير مهمة لأننا لا نتكلم عن نقطة بل عن فترة.

اذا موجود داخل الاحتمال X استعمل الـ Z وذلك نطرح الكل من وسطها على سيغما Z = X-μ/σ

واذا موجود داخل الاحتمال Z فاجد الاحتمال من الجدول مباشرة.

$z < -u$

Example

let x be $N(2, 25)$; find $p(0 < x < 10)$

$$P(0 < x < 10) = p\left(\frac{0-2}{5} < \frac{x-2}{5} < \frac{10-2}{5}\right)$$

$$= p\left(\frac{-2}{5} < z < \frac{8}{5}\right)$$

$$= \phi\left(\frac{8}{5}\right) - \phi\left(\frac{-2}{5}\right)$$

$$= \phi\left(\frac{8}{5}\right) - \left(1 - \phi\left(\frac{2}{5}\right)\right)$$

$$= \phi\left(\frac{8}{5}\right) - 1 + \phi\left(\frac{2}{5}\right)$$

$$= 0.9445 - 1 + 0.655$$

$$= \boxed{0.603}$$

example

let $x \sim N(3, 4)$ Find $p(x \leq 4)$

$$p(x \leq 4) = p\left(\frac{x-\mu}{\sigma} \leq \frac{4-\mu}{\sigma}\right)$$

$$= p\left(\frac{x-\mu}{\sigma} \leq \frac{4-3}{2}\right)$$

$$= p\left(z < \frac{1}{2}\right)$$

$$= \phi\left(\frac{1}{2}\right) = N\left(\frac{1}{2}\right)$$

$$= \boxed{0.6915}$$

le: If x is Normally distributed with mean μ and Variance σ

Show that $P(\mu-\sigma < x < \mu+\sigma) = 0.682$.

where $\phi(1) = 0.841$ from table

$$\mu-\sigma < x < \mu+\sigma) = P\left(\frac{\mu-\sigma-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{\mu+\sigma-\mu}{\sigma}\right)$$

$$= P(-1 < z < 1)$$

$$= \phi(1) - \phi(-1)$$

$$= \phi(1) - (1-\phi(1))$$

$$= \phi(1) - 1 + \phi(1)$$

$$= 2\phi(1) - 1$$

$$= 2(0.841) - 1$$

$$= \boxed{0.682}$$

ple: If x is $N(3, 9)$ find (i) $P(x>0)$, (ii) $P(|x-3|>6)$

Where from the table $\boxed{\phi(1) = 0.8413}$

$$P(x>0) = P\left(\frac{x-\mu}{\sigma} > \frac{0-\mu}{\sigma}\right) = P\left(\frac{x-3}{3} > \frac{0-3}{3}\right)$$

$$= P(z > -1)$$

$$= P(z > -1) = 1 - P(z < -1)$$

$$= 1 - \phi(-1)$$

$$= 1 - (1-\phi(1))$$

$$= 1 - 1 + \phi(1)$$

$$= \phi(1)$$

$$= \boxed{0.8413}$$

$= 1 - (z < -1) = 1 - \phi(-1)$

$= 1 - (1 - \phi(1))$

$= \phi(1)$

$X \sim N(3,9)$

$$P(|X-3|>6) = P(X-3>6) + P(-X+3>6)$$

$$= P(X>9) + P(-X>3)$$

$$= P(X>9) + P(X<-3)$$

$$= P\left(\frac{X-3}{3} > \frac{9-3}{3}\right) + P\left(\frac{X-3}{3} < \frac{-3-3}{3}\right)$$

$$= P(Z>2) + P(Z<-2) = 1-P(Z<2)+P(Z<-2)$$

$$= 1-\phi(2) + \phi(-2)$$

$$= 1-\phi(2) + (1-\phi(2))$$

$$= 1-\phi(2) + 1-\phi(2)$$

$$= 2-2\phi(2)$$

$$= 2(1-\phi(2))$$

from the table $\phi(2) = 0.9772$

$$= 2(1-0.9772)$$

$$= \boxed{0.0456}$$

example If $X \sim N(25, 36)$ Find the Value c (constant) such that

$P(|X-25| \le c) = 0.9544$. where $\phi(2) = 0.9772$.

$$P(|X-25| \le c) = P(X-25 \le c) + P(-X+25 \le c)$$

$$0.9544 = P(X \le c+25) + P(-X \le c-25)$$

$$= P(X \le c+25) + P(X \ge -c+25)$$

$$= P(-c+25 \le X \le c+25)$$

$$= P\left(\frac{-c+25-25}{6} \le \frac{X-25}{6} \le \frac{c+25-25}{6}\right)$$

$$= P\left(\frac{-c}{6} \le Z \le \frac{c}{6}\right)$$

$$= \phi\left(\frac{c}{6}\right) - \phi\left(\frac{-c}{6}\right)$$

$$0.9544 = \phi\left(\frac{c}{6}\right) - 1 + \phi\left(\frac{c}{6}\right)$$

$$0.9544+1 = 2\phi\left(\frac{c}{6}\right)$$

$$\phi\left(\frac{c}{6}\right) = \frac{1.9544}{2}$$

$$\phi\left(\frac{c}{6}\right) = 0.9772 \Rightarrow \frac{c}{6} = 2 \Rightarrow \boxed{c = 12}$$

but $\phi(2) =$ then

ex. 7 let $x \sim N(4, 25)$.

$\Phi(0.4) = 0.6554$

$\Phi(0.2) = 0.5793$

1. Write the p.d.f of x?
2. Find $p(x<6)$?
3. Find $p(3<x<5)$?

Sol. $x \sim N(\mu, \sigma^2) \Longrightarrow f(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} & -\infty<x<\infty \\ 0 & o.w \end{cases}$

$\Longrightarrow$ **1.** $x \sim N(4, 25) \Longrightarrow f(x) = \frac{1}{\sqrt{2\pi(25)}} e^{-\frac{(x-4)^2}{(2)(25)}}$

$$\Longrightarrow \boxed{f(x) = \begin{cases} \frac{1}{\sqrt{50\pi}} e^{-\frac{(x-4)^2}{50}} & -\infty<x<\infty \\ 0 & o.w \end{cases}}$$

2. $p(x<6) = p\left(\frac{x-\mu}{\sigma} < \frac{6-\mu}{\sigma}\right)$ $\qquad z = \frac{x-\mu}{\sigma}$

$= p\left(z < \frac{6-4}{5}\right)$

$= p\left(z < \frac{2}{5}\right) = p(z < 0.4) = \Phi(0.4) = 0.65$

3. $p(3<x<5) = p\left(\frac{3-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{5-\mu}{\sigma}\right)$

$=$

$= p\left(\frac{3-4}{5} < z < \frac{5-4}{5}\right) = p(-0.2 < z < 0.2)$

$= \Phi(0.2) - \Phi(-0.2) = \Phi(2) - [1 - \Phi(0.2)]$

(13) سنأخذ كتاب الوقت فإذا فرض ان كمية الوقت التي تشاهد فيه شاهدة هاتف
كل فترة زمنية معينة فسنأخذ توزيع كاما. وهكذا يعتبر التوزيعات المهمة
وصف هذا التوزيع نوعين توزيعات مهمة أخرى

5-2-3 Gamma Distribution

$X \sim Gamma(\alpha, \lambda)$

A r.v. X is said to have a gamma distribution with parameters α and λ, if it's p.d.f is given by:

$$f(x) = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma\alpha} x^{\alpha-1} e^{-\lambda x} & x > 0 \\ 0 & o.w \end{cases}$$

$\alpha, \lambda > 0$

$$\Gamma\alpha = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx$$

Gamma function

$$\Gamma\alpha = (\alpha - 1)!$$

1.

NOTE: If $\alpha = 1$, X is said to have an exponent distribution,

$$f(x) = \frac{\lambda^1}{\Gamma 1} x^{1-1} e^{-\lambda x}$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & o.w \end{cases}$$

$\boxed{2.}$ If $k = \frac{n}{2}$, $\lambda = \frac{1}{2}$, x is said to have an chi-square distribution $\chi^2(n)$:

$$f(x) = \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma\frac{n}{2}} x^{\frac{n}{2}-1} e^{-\frac{1}{2}x} \qquad x > 0$$

$$\Longrightarrow \boxed{f(x) = \begin{cases} \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} x^{\frac{n}{2}-1} e^{-\frac{1}{2}x} & x > 0 \\ 0 & o.w \end{cases}}$$

$\boxed{ex.9}$ let $x \sim Gamma(k=2, \lambda=4)$.

1. Write the p.d.f of x?
2. Find $p(x < 3)$?

Sol. $f(x) = \frac{\lambda^k}{\Gamma k} x^{k-1} e^{-\lambda x} \qquad x > 0$

$\boxed{1.}$

$$\Longrightarrow f(x) = \frac{4^2}{\Gamma 2} x^{2-1} e^{-4x} \Longrightarrow \Gamma 2 = (2-1)! = 1! =$$

$$\Longrightarrow \boxed{f(x) = \begin{cases} 16\, x\, e^{-4x} & x > 0 \\ 0 & o.w \end{cases}}$$

$\boxed{2.}$ $p(x<3) = \int_0^3 16\, x\, e^{-4x}\, dx$

$\boxed{ex.10}$ let x be r.v has the following p.d.f:

$$f(x) = \begin{cases} A\, x^2 e^{-5x} & x>0 \\ 0 & o.w \end{cases}$$

Find the value of A, what is the name of the distribution

sol.

$$f(x) = \frac{\beta^{\alpha}}{\Gamma\alpha}\, x^{\alpha-1}\, e^{-\beta x} \qquad x>0$$

$$f(x) = A\, x^2\, e^{-5x}$$

$$\Longrightarrow \beta = 5, \quad \alpha = 3$$

$$\therefore A = \frac{\beta^{\alpha}}{\Gamma\alpha} = \frac{5^3}{\Gamma 3} = \frac{125}{2!} = \frac{125}{2}$$

$$\Longrightarrow \boxed{f(x) = \begin{cases} \frac{125}{2}\, x^2\, e^{-5x} & x>0 \\ 0 & o.w \end{cases}}$$

$\therefore x \sim Gamma(\alpha = 3, \beta = 5)$

5.2-4 $\boxed{\text{Exponential distribution}}$ $\boxed{X \sim \exp(\theta)}$

A r.v. X is said to have an exponential distribution if it's p.d.f is given by:

$$\boxed{f(x) = \begin{cases} \theta e^{-\theta x} & x > 0 \\ 0 & o.w \end{cases}} \quad , \theta > 0$$

The c.d.f F:

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\theta x} & 0 \leq x < \infty \\ 1 & x \to \infty \end{cases}$$

proof:

$$F(x) = p(X \leq x) = \int_{-\infty}^{x} f(u)\,du = \int_{0}^{x} \theta e^{-\theta u}\,du$$

$$= -\int_{0}^{x} -\theta e^{-\theta u}\,du = -\left[e^{-\theta u}\right]_{0}^{x}$$

$$= -\left[e^{-\theta x} - e^{-\theta(0)}\right] = -\left[e^{-\theta x} - 1\right]$$

$$= 1 - e^{-\theta x}$$

Exponential

ex 12 let $x \sim \exp(2)$. **a.** Write the p.d.f of x?

b. Find the c.d.f of x?

c. Find $p(x>2)$, $p(3<x<5)$ & $p(|x|<3)$?

a $f(x) = \begin{cases} \theta e^{-\theta x} & 0<x<\infty \\ 0 & o.w \end{cases}$ $\quad x \sim \exp(2)$ then

$f(x) = \begin{cases} 2e^{-2x} & 0<x<\infty \\ 0 & o.w. \end{cases}$ The p.d.f of x.

b c.d.f. $F(x) = \begin{cases} 0 & x \le 0 \\ 1-e^{-\theta x} & 0<x<\infty \\ 1 & x \to \infty \end{cases}$ but $x \sim Exp(2)$ then

$F(x) = \begin{cases} 0 & x \le 0 \\ 1-e^{-2x} & 0<x<\infty \\ 1 & x \to \infty \end{cases}$ the c.d.f.

ex $F(x) = \int_0^x f(t)dt = -\int_0^x -\theta e^{-\theta t} dt = -\left[e^{-\theta t}\right]_0^x = -\left[e^{-\theta x} - 1\right] = 1 - e^{-\theta x}$

$\Rightarrow F(x) = \begin{cases} 0 & x \le 0 \\ 1-e^{-\theta x} & 0<x<\infty \\ 1 & x \to \infty \end{cases}$

c (i) $p(x>2) = 1 - p(x \le 2) = 1 - \int_0^2 f(x)dx = 1 - (-)\int_0^2 -2e^{-2x} dx = 1 + \left(e^{-2x}\right]_0^2) = 1 + (e^{-4} - 1)$

$= \cancel{1} + e^{-4} - \cancel{1} = \boxed{e^{-4}}$

c (ii) $p(3<x<5) = \int_3^5 f(x)dx = -\int_3^5 2e^{-2x} dx = -\left[e^{-2x}\right]_3^5 = -\left[e^{-10} - e^{-6}\right] = \boxed{e^{-6} - e^{-10}}$

c (iii) $p(|x|<3) = p(-3<x<+3) = p(0<x>-3) + p(0<x<3) = 0 + (-)\int_0^3 -2e^{-2x} dx = -\left[e^{-2x}\right]_0^3 = -(e^{-6} - 1) = \boxed{1 - e^{-6}}$

or $= 1 - p(x<3) = 1 - p(0<x<3) = 1 - [F(3) - F(0)] = 1 - [1 - e^{-6} - \cancel{1}] = \cancel{1} - \cancel{1} + e^{-6} = \boxed{1 - e^{-6}}$

or By using c.d.f. (i) $p(x>2) = p(2<x<\infty) = F(\infty) - F(2) = 1 - (1 - e^{-4}) = \cancel{1} - \cancel{1} + e^{-4} = e^{-4}$

(ii) $p(3<x<5) = F(5) - F(3) = 1 - e^{-10} - (1 - e^{-6}) = \cancel{1} - e^{-10} - \cancel{1} + e^{-6} = \boxed{e^{-6} - e^{-10}}$

(iii) $p(|x|<3) = p(x<3) + p(-x<3) = p(x<3) + p(x>-3)$

$= p(-3<x<3) = F(3) - F(-3) = 0 - (1 - e^{-6}) = \boxed{e^{-6} - 1}$

(iv) $p(x<2) = F(2) - F(0) = 1 - e^{-(2)} - (1 - e^{0}) = 1 - e^{-6}$

5.2_5 **Beta Distribution** $\boxed{X \sim Beta(a,b)}$

A r.v. X is said to have a Beta distribution if a p.d.f is given by:

$$\boxed{f(x) = \begin{cases} \dfrac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ 0 & o.w \end{cases}} \quad a,b > 0$$

where $\boxed{B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx}$

$$\boxed{B(a,b) = \frac{\Gamma a \, \Gamma b}{\Gamma a+b}}$$ where $\Gamma a = (a-1)!$, $\Gamma b = (b-1)!$

Note: If $a = b = 1$, then X is distributed uniform distribution on the interval $(0,1)$:

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \quad a = b = 1$$

$$\Rightarrow f(x) = \frac{1}{B(1,1)} x^{1-1}(1-x)^{1-1} = \frac{1}{\frac{\Gamma 1 \, \Gamma 1}{\Gamma 2}} x^0 (1-x)^0$$

$$= \frac{\Gamma 2}{\Gamma 1 \, \Gamma 1} = \frac{1!}{0! \, 0!} = \frac{1}{1} = 1$$

$$\Rightarrow f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & o.w \end{cases}$$ uniform dist

$\boxed{A = 60}$

Beta.

Example. The percentage of betters who make money of the race on a given day has beta distribution with parameter $a=1$, $b=5$. Find the prob. that fewer than ten percent comment winner on given day?

$P(X < 0.1)$.

$\frac{10}{100} = 0.1$

Sol.

P(fewer than 10 percent comment winner on given day)

$= P(X < 0.1)$

$X \sim Beta(a,b) \Rightarrow X \sim B(1,5)$

$X \sim B(a,b)$

$$\therefore f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, & 0 < x < 1 \\ 0 & o.w \end{cases}$$

$\frac{1}{B(a,b)} = \frac{\Gamma(a+b)}{\Gamma a \, \Gamma b}$

$$\therefore f(x) = \begin{cases} \frac{\Gamma(1+5)}{\Gamma 1 \, \Gamma 5} \cdot x^{1-1}(1-x)^{5-1}, & 0 < x < 1 \\ 0 & o.w \end{cases}$$

$$f(x) = \begin{cases} \frac{\Gamma 6}{\Gamma 1 \, \Gamma 5} \cdot x^{0}(1-x)^{4}, & 0 < x < 1 \\ 0 & o.w \end{cases}$$

$$P(X < 0.1) = P(0 < X < 0.1) = \int_0^{0.1} f(x)\,dx$$

$$= \int_0^{0.1} \frac{\Gamma 6}{\Gamma 1 \, \Gamma 5} \cdot (1-x)^4 \, dx = \frac{\Gamma 6}{\Gamma 1 \, \Gamma 5} \left(\frac{(1-x)^5}{5} \Big|_0^{0.1} \right)$$

$$= \frac{5!}{0!\,4!} \frac{1}{5} \left((1-0.1)^5 - (1-0)^5 \right) = \frac{5 \cdot 4!}{4!} (0.9)^5 - (1)^5 = (0.9)^5 - (1)$$

$$= (0.9)^5 - 1$$

Beta

(20)

ex. 14 let x be r.v. having the following p.d.f:

$$f(x) = \begin{cases} A\, x^3 (1-x)^2 & 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$$

1. Find the value of A ?
2. what is the name of the distribution ?
3. Find the c.d.f of x ?
4. Find $p(0 < x < \frac{1}{2})$, $p(x < 1)$, $p(\frac{1}{2} < x < 3)$?

(1) $f(x)$ is a p.d.f. then $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$\Rightarrow \int_0^1 A x^3 (1-x)^2\, dx = 1$

$A \int_0^1 x^3 (1-x)^2\, dx = 1$ we have $B(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1}\, dx$.

then. $a = 4 \rightarrow a-1 = 3$ and $b = 3 \rightarrow b-1 = 2$

$\boxed{a = 3+1 = 4}$ $\boxed{b = 2+1 = 3}$

Then $\boxed{B(a,b) = B(4,3) = \int_0^1 x^{4-1} (1-x)^{3-1}\, dx}$

وبالتعويض عكسها

$A\ B(4,3) = 1$. But $B(a,b) = \frac{\Gamma a\, \Gamma b}{\Gamma a+b}$

$A\ \frac{\Gamma 4 \cdot \Gamma 3}{\Gamma 4+3} = 1$.

$\Gamma r = (r-1)!$

$A\ \frac{(4-1)!\,(3-1)!}{(7-1)!} = 1$

$A \left(\frac{3!\,2!}{6!}\right) = 1$

$A \left(\frac{3!(2!)}{6 \cdot 5 \cdot 4 \cdot 3!}\right) = 1 \Rightarrow A\left(\frac{1}{60}\right) = 1$

$A \frac{1}{60} = 1 \Rightarrow \boxed{A = 60}$

$\boxed{A = 60}$

120

Exersies chapter - 6

Q 1 : Show that the function of the Uniform distribution satisfies the condition of the p. d . f . ?

Q 2 : Show that the function of the Exponential distribution satisfies the condition of the p. d . f . ?

Q 3 : Show that the function of the Gamma distribution satisfies the condition of the p. d . f . ?

Q 4 : Let $X \sim U(2,4)$, : (I) Write the p . d . f . of X ? (II) Find the C. d . f . of X ?
(III) $P(1 < X < 2)$, $P(X > 2)$, $P(2 < X < 3)$, $P(X < 4)$, $P(|X| < 3)$?

Q 5 : Let X be r.v. having the following p . d . f . : $f(x) = \begin{cases} 4e^{-4X} & X > 0 \\ 0 & o.w. \end{cases}$

(I) What is the name of the distribution ?

(II) Find the C . d . f . of X ? (III) Find $P(2 < X < 4)$?

Q 6 : Let X be r.v. having the following p . d . f . : $f(x) = \begin{cases} Ce^{-5X} & X > 0 \\ 0 & o.w. \end{cases}$

(I) Find the value of C and What is the name of the distribution ?

(II) Write the C . d . f . of X ? (III) Find $P(3 < X < 4)$, $P(X > 0)$? , $P(0 < X < 1)$

Q 7 : Let X be N (2 , 25) , Find $p(0 < X < 10)$?

Q 8 : Let X be N (3 , 4) , (I) Write the p.d.f. of X ? (II) Find $P(X \leq 3)$?

Q 9 : If $X \sim N(3,9)$, Find (I) $p(X > 0)$? (II) $p(|X - 3| > 6)$?

Q 10 : If X is Normally distribution with mean (μ) and variance (σ^2) Show that :

$$P(\mu - \sigma \leq X \leq \mu + \sigma) = 0.682$$

Q 11 : Let X be r.v. having the following p . d . f . : $f(x) = \begin{cases} C\, e^{-\frac{(x-3)^2}{8}} & X \in R \\ 0 & o.w. \end{cases}$

(I) Find the value of C and What is the name of the distribution ?

(II) Find $P(2 < X < 5)$, $P(|X| > 3)$?

Q12 : If X ~N(25,36) , Find the constant C such that : $P(|X - 25| \le C) = 0.9544$

Where Ø(2)=0.9772

Q 13 : Let we have the following p . d . f . : $f(x) = \begin{cases} A\ X^4\, e^{-\frac{x}{2}} & X > 0 \\ 0 & o.w. \end{cases}$

(I) What is the value of A ? (II) What is the name of the distribution ?

Ø(0.4) = 0.685 , Ø(0.5) = 0.1915 , Ø(1) = 0.8413 , Ø(2) = 0.9772 , Ø(1.6) = 0.945

Q 14 : Suppose the weights of 1000 babies at birth are normally distributed with mean 8.5 pounds and standard deviation 0.5 pounds . Find the number of babies with weights ? (I) : Less than 9.8 pounds (II) : between 7.2 pounds and 8.5 pounds ? Given that Ø(0) = 0.5 , Ø(2.6) = 0.9938

Q 15 : Let X be Uniformly distribution in the interval (-2 , 2) , Find (I) , p (X < 1) ?

(II) $p(|x - 1| \ge \frac{1}{2})$?

Q 16 : Let X r.v. represent the working time of the certain machine (in hours) with the following p.d.f. :

$f(x) = \begin{cases} \frac{100}{x^2} & , X \ge 100 \\ 0 & o.w. \end{cases}$ Find : (I) The C . d . f . of the r . v . X ? (II) Find th

probability that the machine is working more than 150 hours ?

Q 17 : Let $X \sim N(\mu, \sigma^2)$, Show that : (I) : $p(\mu - 2\sigma < X < \mu + 2\sigma) = 0.954$?

(II) : $p(\mu - 3\sigma < X < \mu + 3\sigma) = 0.999$? Where Ø(2) = 0.9772 , Ø(3) = 0.999

Q 18 : If X ~N(4 , 25) , (I) Write the p.d.f. of X ? (II) Find p (X < 6) ? (III) Find p (3 < X < 6)

Q 20 : Let X be r.v. having the following p . d . f . : $f(x) = \begin{cases} A\,x^2(1-x)^2 & 0 \le x \le 1 \\ 0 & O.W \end{cases}$

Find : (I) : the value of A ? (II) What is the name of the distribution ? (III) the C . d . f . of X ?

(IV) : $p(0 < X < 1/2)$, $p(X < 1)$, $p(\frac{1}{2} < X < 3)$?

Q 19 : Let $X \sim \exp(2)$: (I) : Write the p . d . f . of X ? (II) : Find the C . d . f . of X ?

(III) : Find $p(x > 2)$, $p(3 < x < 5)$, $P(|X| < 3)$

Q 21 : Let X is Uniformly distribution over (0 , 1) calculate the probability that :

(I) : $p(x > 3)$, ? $p(x > 6)$, $p(3 < x < 8)$? (II) : Find the distribution function ?

Q 22 : Suppose the temperature in Iraq during July is normal distributed with mean 45 °C and

standard deviation $\sigma = 2.5$ °C Find the probability that the temperature is :

(I) : between 46.7 C° and 48.1 C° (II) : higher than 47.5 C°

Q 23 : Let X be r.v. with the standard normal distributed Find : (I) $P(0 \le X \le 1.82)$?

(II) $P(-0.73 \le X \le 0)$? (III) $P(-0.37 \le X \le 2.01)$? (IV) $P(0.65 \le X \le 1.86)$?

(V) $P(-1.79 \le X \le -0.54)$? (VI) $P(X \ge 1.13)$? (VII) $P(|X| \le 0.5)$?

Q 24 : Let X ~N(3,9) , Find $P(X^2 \le 9)$ Given that Ø(0) = 0.5 , Ø(1) = 0.8413 , Ø(2) = 0.9772

Q 26 : Write the p.d.f. of the following : (I) X ~ N (2 , 1) ? (II) X ~ Gamma (2 , 1)? (III) X ~ U (2 , 4) ?
(IV) X ~ Beta (5, ½) ? (V) X ~ Exp(2) ? (VI) X ~ χ^2 (3) ?

Q 25 : Let X ~N(105 , 100) , Compute $P(100 < X < 114)$?Given Ø(0.9) = 0.3159 Ø(0.5)= 0.6915

Q 27 : Let X ~N(105 , 100) , Let the space $S = \{X : 0 < X < \infty\}$ and Compute ?
Given That Ø(0.9)=0.3159 , Ø(0.5)=0.6915

Q28 : If X ~N(67,49) , Find the constant C such that : $P(|X - 67| \le C) = 0.9544$

Where Ø(2)=0.9772

Q23: let the space $S = \{X : 0 < X < \infty\}$ and $\boxed{f(x) = e^{-x}}$, $x > 0$, ; A we have $A = \{x : 0 < x < 1\}$
what is $p(X \in A)$? find the c.d.f. of X ? use the c.d.f. to find $p(X \in A)$?