Chapter one

1.1 Introduction

In our day to day life, we perform many activities which have a fixed result no matter any number of times they are repeated. For an example given any triangle, without knowing the three angles, we can definitely say that the sum of measure of angles is 180°.

1.1.1. Random Experiments

We also perform many experimental activities, where the result may not be same, when they are repeated under identical conditions. For example, when a coin is tossed it may turn up a head or a tail, but we are not sure which one of these results will actually be obtained. Such experiments are called random experiments. An experiment is called random experiment if it satisfies the following two conditions:

1. It has more than one possible outcome.
2. It is not possible to predict the outcome in advance.

Check whether the experiment of tossing a die is random or not?

1.1.2 Sample Space

The sample space is the set of all possible outcomes of an experiment.

1.1.3 Event

An event is a subset of the sample, events can be classified into several types: Sure events, impossible events, complement events, mutually exclusive events, independent events.

1.2 Probability

Probability theory is the branch of mathematics that studies the possible outcomes of given events together with the outcomes' relative likelihoods and distributions. In common usage, the word \probability" is used to mean the chance that a particular event (or set of events) will occur expressed on a linear scale from 0 (impossibility) to 1 (certainty), also expressed as a percentage between 0 and 100%. The analysis of data (possibly generated
by probability models) is called statistics. Probability is a way of summarizing the uncertainty of statements or events. It gives a numerical measure for the degree of certainty (or degree of uncertainty) of the occurrence of an event. Another way to define probability is the ratio of the number of favorable outcomes to the total number of all possible outcomes. This is true if the outcomes are assumed to be equally likely. The collection of all possible outcomes is called the sample space. If there are n total possible outcomes in a sample space S, and m of those are favorable for an event E, then the probability of event E is given as:

$$\text{Probability of Event (E)} = \frac{\text{number of event outcomes}}{\text{total number of possible outcomes}}$$

$$\mathrm{P(E)} = \frac{\mathrm{n(E)}}{\mathrm{n(S)}}$$

Ex1: When we flip a coin the sample space is

$$S = \{H, T\}$$

Where (H) denotes that the coin lands Heads and (T) denotes that the coin lands Tails.

For a fair coin we expect (H) and (T) to have the same chance of occurring, i.e., if we flip the coin many times then about 50 % of the outcomes will be (H). We say that the probability of (H) to occur is 0.5 (or 50 %). The probability of (T) to occur is then also 0.5.

Ex2: When we roll a fair die then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

If E represents the event that dies lands with the number 1, then the number of possible cases for E: $n(E) = 1$, the probability of E is:

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}$$

Now if E is the die lands with an even number, then the probability that the die lands with an even number is

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

Ex3: Two coins are tossed together, find the following probabilities:

1. The appearance of one Head (E_1)

2. The appearance of two Heads (E_2)

3. The appearance of three Heads (E_3)

4. The appearance of Head (E_4)

5. there is no Head (E_5)

Sol:

$S = \{HH, HT, TH, TT\}, \quad n(S) = 4$

1. $E_1 = \{HT, TH\}$

$n(E_1) = 2$

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

2. $E_2 = \{HH\}$

$n(E_2) = 1$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{1}{4}$$

3. $E_3 = \{\ \}$

$n(E_3) = 0$

$$P(E_3) = \frac{n(E_3)}{n(S)} = 0$$

4. $E_4 = \{HH, HT, TH\}$

$n(E_4) = 3$

$$P(E_4) = \frac{n(E_4)}{n(S)} = \frac{3}{4}$$

5. $E_5 = \{TT\}$

$n(E_5) = 1$

$$P(E_5) = \frac{n(E_5)}{n(S)} = \frac{1}{4}$$

Ex4: A coin is tossing thrice, find the following probabilities:

1. The appearance of one Head (E_1)
2. The appearance of two Heads (E_2)
3. The appearance of three Heads (E_3)
4. The appearance of Head (E_4)
5. there is no Head (E_5)

Sol:

$S = \{HHH, HHT, HTH, THH\ HTT, THT, TTH, TTT\ \}, \quad n(S) = 8$

1. $E_1 = \{HTT, THT, TTH\}$

$n(E_1) = 2$

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{3}{8}$$

2. $E_2 = \{HHT, HTH, THH\ \}$

$n(E_2) = 3$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{3}{8}$$

3. $E_3 = \{HHH\}$

$n(E_3) = 1$

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{1}{8}$$

4. $E_4 = \{HHH, HHT, HTH, THH\ HTT, THT, TTH\}$

$nE_4 = 7\ \ P(E_4)$

$$P(E_4) = \frac{n(E_4)}{n(S)} = \frac{7}{8}$$

5. $E_5 = \{TTT\}$

$n(E_5) = 1$

$$P(E_5) = \frac{n(E_5)}{n(S)} = \frac{1}{8}$$

Ex5: Two dice are thrown together, find the following probabilities:

1. The number shown is 1

2. The numbers shown are 1 or 4

3. The two numbers shown are 1 and 4

4. The sum of the two numbers shown is 6.

5. The sum of the two numbers shown is more than 9.

6. The sum of the two numbers shown is 10 or more.

7. The quotient of the two numbers that appears is greater than 3

8. The Subtraction of the two numbers shown is greater than 3

Sol:

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$$
$$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$$
$$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)$$
$$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$n(S) = 36$

1. $E_1 = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)\}$

$n(E_1) = 11$

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{11}{36}$$

2. $E_2 = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)$

$,(2,4),(3,4),(5,4),(6,4),(4,2),(4,3),(4,4),(4,5),(4,6)\}$

$n(E_2) = 20$

$$P(E_2) = \frac{n(E_2)}{n(S)} = \frac{20}{36} = \frac{5}{9}$$

3. $E_3 = \{(1,4),(4,1)\}$

$n(E_3) = 2$

$$P(E_3) = \frac{n(E_3)}{n(S)} = \frac{2}{36} = \frac{1}{18}$$

4. $E_4 = \left\{\{(1,5),(2,4),(3,3),(4,2),(5,1)\right\}$

$n(E_4) = 5$

$$P(E_4) = \frac{n(E_4)}{n(S)} = \frac{5}{36}$$

5. $E_5 = \{(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\}$

$n(E_5) = 6$

$$P(E_5) = \frac{n(E_5)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

6. $E_6 = \{(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\}$

$n(E_6) = 6$

$$P(E_6) = \frac{n(E_6)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

7. $E_7 = \{(1,4),(1,5),(1,6),(4,1),(5,1),(6,1)\}$

$n(E_7) = 6$

$$P(E_7) = \frac{n(E_7)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

8. $E_8 = \{(1,5),(1,6),(2,6),(5,1),(6,1)\}$

$n(E_8) = 6$

$$P(E_8) = \frac{n(E_8)}{n(S)} = \frac{5}{36}$$

1.2.1 Basic Concepts of probabilities:

There are four basic probability rules, these rules are helpful in solving probability problems, in understanding the nature of probability, and in deciding if your answers to the problems are correct.

Probability Rule 1

The probability of any event E is a number (either a fraction or decimal) between and including 0 and 1. This is denoted by $0 \leq P(E) \leq 1$.

Probability Rule 2

If an event E cannot occur (i.e., the event contains no members in the sample space), its probability is 0.

Probability Rule 3

If an event E is certain, then the probability of E is 1.

Probability Rule 4

The sum of the probabilities of all the outcomes in the sample space is 1.

1.2.2 Types of Event:

Now it is necessary to give a definition of the classes of events and some other necessary terms:

- **Sure event**: It is an event which probability of occurrence is certain, and in a mathematical sense $P(E) = 1$.

- **Impossible event**: If E is an impossible event, then $P(E) = 0$.

- **Simple Event:**

  If an event E has only one sample point of a sample space, it is called a simple (or elementary) event. In a sample space containing (n) distinct elements, there are exactly n simple events.

For example, in the experiment of tossing a coin twice, a sample space is $S = \{HH, HT, TH, TT\}$. There are four simple events corresponding to this sample space, these are $E_1 = \{HH\}, E_2 = \{HT\}, E_3 = \{TH\}, E_4 = \{TT\}$.

- **Compound Event:** If an event has more than one sample point, it is called a Compound event. For example, in the experiment of tossing a coin thrice the events

  E: Exactly one head appeared

  F: At least one head appeared

  G: At most one head appeared.

  are all compound events. The subsets of S associated with these events are

  $E = \{HTT, THT, TTH\}$

  $F = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}$

  $G = \{TTT, THT, HTT, TTH\}$

Each of the above subsets contain more than one sample point, hence they are all compound events.

- **Mutually exclusive events:** These are events that cannot happen together**.** In general, two events E_1 and E_2 are called mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event, i.e., if they cannot occur simultaneously. In this case the sets E_1 and E_2 are disjoint, that is $P(E_1 \cap E_2) = 0$

Remark: Simple events of a sample space are always mutually exclusive.

- **Exhaustive (universal) events:** Consider the experiment of throwing a die, we have $S = \{1, 2, 3, 4, 5, 6\}$. Let us define the following events

A: a number less than 3 appears,

B: a number greater than 2 but less than 5 appears

C: a number greater than 4 appears.

Then $A = \{1, 2\}, B = \{3,4\}, C = \{5, 6\}$. We observe that

$A \cup B \cup C = \{1, 2\} \cup \{3, 4\} \cup \{5, 6\} = S.$

$P(A \cup B \cup C) = 1$

Such events A, B and C are called exhaustive or universal events.

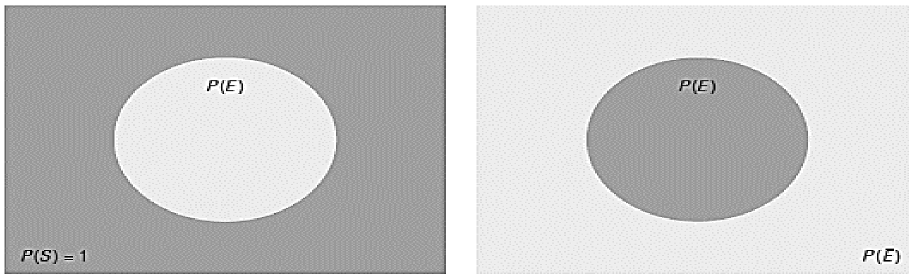
In general, if $E_1, E_2, \ldots, E_n$ are n events of a sample space S and if

$$E_1 \cup E_2 \cup \ldots \cup E_n = \bigcup_{i=1}^{n} E_i = S$$

Then $E_1, E_2, \ldots, E_n$ are called exhaustive events. In other words, events $E_1, E_2, \ldots, E_n$ are said to be exhaustive if at least one of them necessarily occurs whenever the experiment is performed.

- **Independent events:** These are events that if one of them occurs does not affect the other, thus the probability for independent events is given as $P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1)\, P(E_2) \ldots P(E_n)$

- **Complement event**: E is the set of outcomes in the sample space, then complement event of E that are not included in the outcomes of event E. The complement of E is denoted by E^C or $\overline{E}$.



(a) Simple probability **(b)** $P(\bar{E}) = 1 - P(E)$

1.2.2 Some properties of probabilities:

The following properties of any two events A and B can be summarized:

1. $P(A^C) = 1 - P(A),\ P(B^C) = 1 - P(B)$

2. $P(A) + P(A^C) = 1\ ,\ P(B) + P(B^C) = 1$

3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

4. $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

5. $P(A \cap B^C) = P(A) - P(A \cap B)$

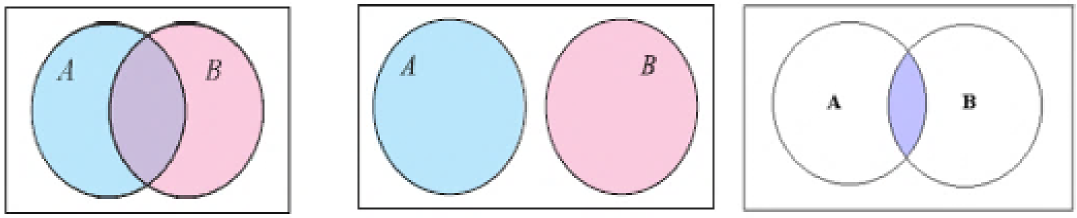
6. $P(A^C \cap B) = P(B) - P(A \cap B)$

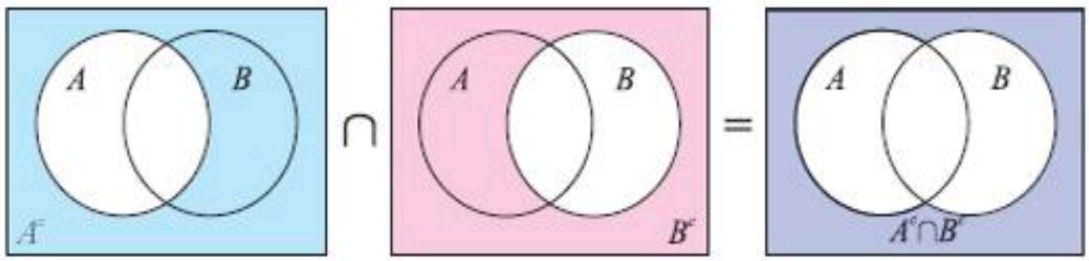
7. $P(A^C \cap B^C) = 1 - P(A \cup B)$

8. $P(A^C \cup B^C) = 1 - P(A \cap B)$

9. $P[(A \cap B)^C] = 1 - P(A \cap B)$

10. $P[(A \cup B)^C] = 1 - P(A \cup B)$

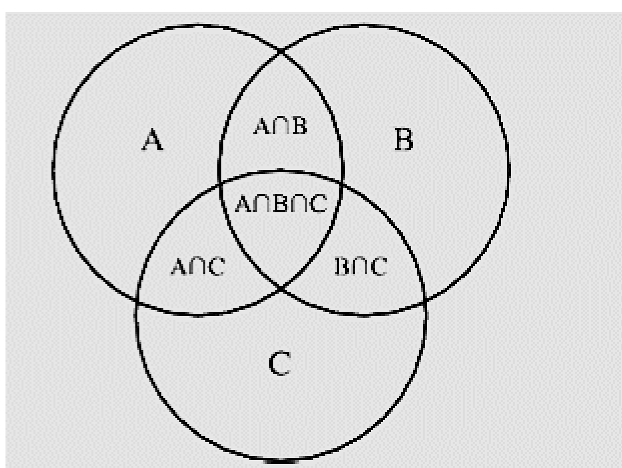




Now for three events A, B and C can be summarized:

11.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



12. $P(A \cup B) \cap C = P(A \cap C) \cup P(B \cap C)$

13. $P(A \cap B) \cup C = P(A \cup C) \cap P(B \cup C)$

Remarks:

- If A and B are **mutually exclusive** events, then:

$$P(A \cup B) = P(A) + P(B)$$

- In probability concept, it is an important to explain terms **(and)**, **(or)**

$$P(A\ and\ B) = \frac{n(A\ and\ b)}{n(S)} = P(A \cap B) = \frac{n(A \cap B)}{n(S)}$$

$$P(A\ or\ B) = \frac{n(A\ or\ B)}{n(S)} = P(A \cup B) = \frac{P(A \cup B)}{n(S)}$$

- Furthermore, we know that $A - B$ is the set of all those elements which are in A but not in B. Therefore, the set $A\text{–}B$ may denote the event ‘A but not B’ or ‘A and not B’. We know that $A - B = A \cap B^C$, so:

$$P(A\ but\ not\ B) = \frac{n(A\ but\ not\ B)}{n(S)} = P(A \cap B^C) = \frac{P(A \cap B^C)}{n(S)}$$

Ex6: Referring to the experiment of tossing a coin twice, let A be the event “at least one head occurs” and B the event “the second toss results in a tail.” Then $A = \{HT, TH, HH\}$, $B = \{HT, TT\}$ and so we have:

$A \cup B = A + B - A \cap B = \{HT, TH, HH, TT\} = S$, where $A \cap B = \{HT\}$.

$A^C = \{TT\}$, $B^C = \{TH. HH\}$

$A - B = \{TH, HH, TT\}$

Therefore,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{4} + \frac{2}{4} - \frac{1}{4} = 1$$