Chapter Three

Conditional Probability and Bayes Rule

3.1 Conditional Probability:

If E and F are two events **associated (joint)** with the same sample space of a random experiment, then the conditional probability of the event E under the condition that the event F has occurred, written as $P(E \mid F)$, is given by:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} \text{ , } P(F) \neq 0$$

It is important to note that conditional probability itself is a probability measure, so it satisfies probability axioms. In particular,

Axiom1: For any events E, F, then $P(E \mid F) \geq 0$

Axiom2: $P(E \cap F) = P(F)P(E \mid F) \text{ , } P(F) \neq 0$

Axiom3: Conditional probability of $P(E \mid E) = 1$

Axiom4: $P(E^C \mid F) = 1 - P(E \mid F)$

Axiom5: $P(F \cap G|E) = P(F|E)\ P(G|E, F)$

$$or\ P(G|E, F) = \frac{P(F \cap G|E)}{P(F|E)}$$

Axiom6: $P(E \cap F \cap G) = P(E)\ P(F|E)\ P(G|E, F)$

$$or\ P(G|E, F) = \frac{P(E \cap F \cap G)}{P(E)\ P(F|E)}$$

Axiom7: $P(E \mid F) \neq P(F \mid E)$

Axiom8: if the events E and F are independents, then

$$P(E \mid F) = \frac{P(E)\ P(F)}{P(F)} = P(E) \text{ , } P(F \mid E) = \frac{P(F)\ P(E)}{P(E)} = P(F)$$

Axiom9: Let E be an event and let $F_1, F_2, \dots, F_n$ be a **disjoint** collection of events for which $P(F_1) > 0$ for all i and such that $E \subset \bigcup_{i=1}^{n} F_i$. Suppose $P(F_i)$ and $P(E|F_i)$ are known. Then $P(E)$ may be computed as:

$$P(E) = \sum_{i=1}^{n} (E|F_i)\, P(F_i) = (E|F_1)P(F_1) + \cdots + (E|F_n)P(F_n)$$

Ex31: Find the probability that a single toss of a die will result in a number less than 4 given an odd number.

Sol:

$S = \{1,2,3,4,5,6\}, \qquad n(S) = 6$

Let E be the event of getting a number less than 4, then:

$E = \{1,2,3\}, \qquad n(E) = 3$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6}$$

Let F be the event of getting an odd number, then:

$F = \{1,3,5\}, \quad n(F) = 3$, thus:

$$P(F) = \frac{n(F)}{n(S)} = \frac{3}{6}$$

$$(E \cap F) = \{1,3\} \Rightarrow P(E \cap F) = \frac{2}{6}$$

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{2/6}{3/6} = \frac{2}{3}$$

Ex32: Suppose you roll two dice, find that probability the sum is 6 given the first die shows a 3.

Sol:

The elements number of the sample space is $n(S) = 36$

Let A is the event that the sum is 6, then:

$$A = \{(1,5), (2,4), (3,3), (5,1), (4,2)\} \Rightarrow P(A) = \frac{5}{36}$$

Let B is the event that the first die shows a 3, then:

$$B = \{(3,1), (3,2), (3,3), (3,4)(3,5), (3,6)\} \Rightarrow P(B) = \frac{6}{36}$$

$$(A \cap B) = \{3,3\} \Rightarrow \ P(A \cap B) = \frac{1}{36}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{6/36} = \frac{1}{6}$$

Ex33: A family has 2 children. Given that one of the children is a boy, what is the probability that the other child is also a boy?

Sol: Let:

B be the event that one child is a boy

A the event that both children are boys

$S = \{bb, bg, gb, gg\}, \qquad n(S) = 4$

$$A = \{bb\} \Rightarrow P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}$$

$$B = \{bb, bg, gb\} \Rightarrow P(B) = \frac{n(B)}{n(S)} = \frac{3}{4}$$

$$(A \cap B) = \{bb\} \Rightarrow \ P(A \cap B) = \frac{1}{4}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Ex34: Suppose a box contain 3 red marbles and 2 black ones. We select 2 marbles. Find the probability that second marble is red given that the first one is red.

Sol: The sample space here represents selecting 2 marbles among 5 marbles, therefore:

$$n(S) = \binom{5}{2} = \frac{5!}{2!\,3!} = 10$$

Let R_1 be the event the second marble is red, and R_2 the event that the first one is red therefore:

$$P(R_2) = \frac{3}{5}$$

while $P(R_1 \mid R_2)$ is the probability both are red, or is the probability that we chose 2 red out of 3 and 0 black out of 2, thus

$$P(R_1 \cap R_2) = \frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}} = \frac{\frac{3!}{2!\,1!}\ \frac{2!}{0!\,2!}}{\frac{5!}{2!\,3!}} = \frac{3}{10}$$

$$P(R_1 \mid R_2) = \frac{P(R_1 \cap R_2)}{P(R_2)} = \frac{3/10}{3/5} = \frac{1}{2}$$

Ex35: The results of a survey of a group of 100 people who bought either a mobile phone or a tablet from any of the two brands Samsung and Nokia is shown in the table below

Brands	Phone	Tablet
Samsung	20	10
Nokia	30	40

If a person is selected at random from the group, what is the probability that

1. bought brand Nokia?
2. bought a mobile phone from brand Nokia?
3. bought a mobile phone given that bought brand Nokia?

Sol:

Let

event M: bought a mobile phone

event T: bought a tablet

event G: bought brand Samsung

event N: bought brand Nokia

1. A total of 70 people out of the total of 100 bought brand N, hence

$$P(N) = \frac{n(N)}{n(S)} = \frac{70}{100} = 0.7$$

2. A total of 30 people out of 100 bought a mobile from brand N

$$P(M\ and\ N) = P(M \cap N) = \frac{n(M \cap N)}{n(S)} = \frac{30}{100} = 0.3$$

3. $P(M|N) = \frac{P(M \cap N)}{P(N)} = \frac{0.3}{0.7} = 0.43$

Ex36: In a group of kids, if one is selected at random the probability that he/she likes oranges is 0.6, the probability that he/she likes oranges AND apples is 0.3. If a kid, who likes oranges, is selected at random, what is the probability that he/she also likes apples?

Sol: Let

O: kid likes oranges.

A: kid likes apples.

Given $P(O) = 0.6, P(A\ and\ O) = 0.3$

the conditional probability $P(A|O)$ that the kid likes apples given that he likes oranges.

$$P(A|O) = \frac{P(A\ and\ O)}{P(O)} = \frac{0.3}{0.6} = 0.5$$

Ex37: Suppose we draw a card from a pack of playing cards. Find the probability of drawing

1. A Queen, given that the card drawn is of Heart.

2. A Queen, given that the card drawn is a Face card.

Sol:

1. $P(Queen|Heart) = \frac{P(Queen \cap Heart)}{P(Heart)}$

$$P(Queen \cap Heart) = \frac{\binom{1}{1}}{\binom{52}{1}} = \frac{1}{52}$$

$$P(Heart) = \frac{\binom{13}{1}}{\binom{52}{1}} = \frac{13}{52}$$

$$P(Queen|Heart) = \frac{1/52}{13/52} = \frac{1}{13}$$

2. $P(Queen|Face) = \frac{P(Queen \cap F)}{P(F)}$

$$P(Queen \cap Face) = \frac{\binom{4}{1}}{\binom{52}{1}} = \frac{4}{52}$$

$$P(Face) = \frac{\binom{12}{1}}{\binom{52}{1}} = \frac{12}{52}$$

$$P(Queen|Face) = \frac{4/52}{12/52} = \frac{1}{3}$$

Ex38: If $P(A) = 0.8, P(B) = 0.5 \; and \; P(B|A) = 0.4.$

Find $P(A \cap B), P(A|B), P(A \cup B)$

Sol:

Since $P(A \cap B) = P(B \cap A)$, then:

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(A \cap B) = P(B|A)P(A) = 0.4 * 0.8 = 0.32$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.32}{0.5} = 0.64$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.8 + 0.5 - 0.32 = 0.98$$

Ex39: If $P(A) = 6/11 \, , P(B) = 5/11 \; and \; P(A \cup B) = 7/11.$

Find $P(A \cap B), P(A|B), \; P(B|A)$

Sol:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$P(A \cap B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{4/11}{5/11} = \frac{4}{5}$$

Since $P(A \cap B) = P(B \cap A)$, then:

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{4/11}{6/11} = \frac{2}{3}$$

Ex40: A coin tossed three times, suppose that:

a) E: head on third toss, F: head on first two tosses.

b) E: at least two heads, F: at most two heads.

c) E: at most two tails, F: at least one tail.

Find $P(E|F)$ and $P(F|E)$ in each case.

Sol:

The sample space S is given as:

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}, n(s) = 8$

a) $E = \{HHH, HTH, THH, TTH\}, n(E) = 4$

$F = \{HHH, HHT\}, n(F) = 2$

$E \cap F = \{HHH\}, n(E \cap F) = 1$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{8}, P(F) = \frac{n(F)}{n(S)} = \frac{2}{8}, P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{1}{8}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/8}{2/8} = \frac{1}{2}$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/8}{4/8} = \frac{1}{4}$$

b) $E = \{HHH, HHT, HTH, THH\}, n(E) = 4$

$F = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}, n(F) = 7$

$E \cap F = \{HHT, HTH, THH\}, n(E \cap F) = 3$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{8}, P(F) = \frac{n(F)}{n(S)} = \frac{7}{8}, P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{3}{8}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{3/8}{7/8} = \frac{3}{7}$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{3/8}{4/8} = \frac{3}{4}$$

c) $E = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}, n(E) = 7$

$F = \{HHT, HTH, THH, HTT, THT, TTH, \}, n(F) = 6$

$$E \cap F = \{HHT, HTH, THH, HTT, THT, TTH\}, n(E \cap F) = 6$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{7}{8}, P(F) = \frac{n(F)}{n(S)} = \frac{6}{8}, P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{6}{8}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{6/8}{7/8} = \frac{6}{7}$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{6/8}{6/8} = 1$$