Chapter Four

Random Variables

4.1 Introduction

A random variable is a function defined on a sample space. The values of the function can be anything at all, but for us they will always be numbers, the random variable is usually denoted by r.v..

Suppose that to each point of a sample space we assign a number. We then have a function defined on the sample space. This function is called a random variable or more precisely a random function. It is usually denoted by a capital letter such as X or Y.

For example if a coin is tossed twice, let X represent the number of heads that can come up, then the sample space is $S = \{HH, HT, TH, TT\}$, for each sample point we can associate a number for X as:

Sample point	TT	HT	TH	HH
$X = x$	$X = 0$	$X = 1$	$X = 1$	$X = 2$
$P(X = x)$	$P(X = 0) = 1/4$	$P(X = 1) = 2/4$		$P(X = 2) = 1/4$

It should be noted that many other random variables could also be defined on this sample space, for example, the square of the number of heads or the number of heads minus the number of tails.

Note that a random variable that takes on a countable number values (integers) is called a discrete random variable, while one which takes uncountable number values is called a continuous random variable.

4.2 Discrete random variable:

A random variable that takes a countable number values, suppose the random variable X be a discrete random variable, then $P(x) = P(X = x)$ is called the probability mass function $(p.m.f)$ of the random variable X. The probability mass function of the discrete random variable X have the following properties:

1. $P(x) = P(X = x) \geq 0$

2. $\sum_{\text{all } x} P(x) = \sum_{\text{all } x} P(X = x) = 1$

Note that the probability mass function of X is called the distribution of X

Ex49: Find the probability distribution of tossing a coin thrice, let X denote the number of Tails.

Sol: Since X is a number which values are defined on the outcomes of a random experiment. Therefore, X is a random variable. Now, sample space of the experiment is

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

S.P.	HHH	THH	HTH	HHT	TTH	THT	HTT	TTT
$X = x$	$X = 0$	$X = 1$			$X = 2$			$X = 3$
$P(x)$	$1/8$	$3/8$			$3/8$			$1/8$

$$\sum_{\text{all } x} P(x) = \sum_{i=0}^{3} P(x_i) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$\sum_{\text{all } x} P(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

Ex50: let Y be a random variable with the following probability mass function

Y = y	-1	0	2	3	6
P(Y)	0.2	0.1	0.4	0.1	0.2

Find

1. $P(Y \geq -1)$
2. $P(Y < -1)$
3. $P(Y = 6)$

Sol:

1. $P(Y \geq -1) = \sum_{\text{all } y} P(y) = P(y = -1) + P(y = 0) + P(y = 2) + P(y = 3) + P(y = 6)$

$P(Y \geq -1) = 0.2 + 0.1 + 0.4 + 0.1 + 0.2 = 1$

2. $P(Y < -1) = 0$

3. $P(Y = 6) = 0.2$

Ex51: let X be a random variable with following probability mass function Find

X = x	1	2	3	4	5
P(x)	2c	c	3c	5c	4c

1. The value of c.
2. $P(X \geq 2)$
3. $P(X < 3)$
4. $P(2 < X < 5)$
5. $P(3 \leq X < 4)$

Sol:

1. $\sum_{\text{all x}} P(x) = 1$

$$\sum_{x=1}^{5} P(x) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = 1$$

$$2c + c + 3c + 5c + 4c = 1 \Rightarrow 15c = 1 \Rightarrow c = \frac{1}{15}$$

X = x	1	2	3	4	5
P(x)	2/15	1/15	3/15	5/15	4/15

2. $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$P(X \geq 2) = \frac{1}{15} + \frac{3}{15} + \frac{5}{15} + \frac{4}{15} = \frac{13}{15}$$

or $P(X \geq 2) = 1 - P(X < 2) = 1 - P(X = 1) = 1 - \frac{2}{15} = \frac{13}{15}$

3. $P(X < 3) = P(X = 2) + P(X = 1) = \frac{1}{15} + \frac{2}{15} = \frac{3}{15}$

or $P(X < 3) = 1 - P(X \geq 3) = 1 - P(X = 1)$

$$1 - [P(X = 3) + P(X = 4) + P(X = 5)] = 1 - \frac{12}{15} = \frac{3}{15}$$

4. $P(2 < X < 5) = P(X = 3) + P(X = 4) = \frac{3}{15} + \frac{5}{15} = \frac{8}{15}$

5. $P(3 \leq X < 4) = P(X = 3) = \frac{3}{15}$

4.3 Mathematical expectation of the discrete random variable:

The mathematical expectation for the discrete random variable X is the mean of the probability mass function (distribution), it is denoted by E(X) and can be obtained as:

$$E(X) \text{ or } \mu = \sum_{\text{iall x}} x\, P(x)$$

The mathematical expectation for the discrete random variable X has some characteristics (properties):

1. $E(X^r) = \sum_{\text{all x}} x^r\, P(x)$
2. $E(k) = k$, where k is constant
3. $E(kX) = k\, E(X)$, where k is constant

Ex52: Let X be a random variable with the following p.m.f:

X = x	1	2	4
P(x)	2/10	3/10	5/10

Find:

$E(X)$, $E(X^2)$, $E(\sqrt{X})$, $E(5X)$, $E(2X-2)$

Sol:

$$E(X) = \sum_{\text{all x}} x\, P(x) = (1)\left(\frac{2}{10}\right) + (2)\left(\frac{3}{10}\right) + (4)\left(\frac{5}{10}\right) = \frac{28}{10}$$

$$E(X^2) = \sum_{\text{all x}} x^2\, P(x) = (1)^2\left(\frac{2}{10}\right) + (2)^2\left(\frac{3}{10}\right) + (4)^2\left(\frac{5}{10}\right) = \frac{94}{10}$$

$$E(\sqrt{X}) = \sum_{\text{all } x} \sqrt{x}\, P(x) = \sqrt{1}\left(\frac{2}{10}\right) + \sqrt{2}\left(\frac{3}{10}\right) + \sqrt{3}\left(\frac{5}{10}\right)$$

$$E(5X) = 5\, E(X) = 5\, \frac{28}{10} = 14$$

$$E(2X - 2) = 2\, E(X) - 2 = (2)\, \frac{28}{10} - 2 = \frac{76}{10}$$

4.4 Variance of the discrete random variable:

Let X be a discrete random variable, the variance of X is given as:

$\text{Var}(X)$ or $\sigma_X^2 = E(X^2) - [E(X)]^2$

Where:

$$E(X) = \sum_{\text{all } x} x\, P(x) \quad \text{and} \quad E(X^2) = \sum_{\text{all } x} x^2\, P(x)$$

Also the standard deviation of X is given as:

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{E(X^2) - [E(X)]^2}$$

The variance of the discrete random variable X has some characteristics:

1. $\text{Var}(k) = 0$, where k is constant
2. $E(kX) = k^2\, E(X)$, where k is constant

Ex53: Let X be a random variable with the following p.m.f:

X = x	1	2	4
P(x)	2/10	3/10	5/10

Find:

$E(5X), \text{Var}(X), \text{Var}(2X), \text{Var}(2X - 1)$

Sol:

$$E(X) = \sum_{\text{all x}} x\, P(x) = (1)\left(\frac{2}{10}\right) + (2)\left(\frac{3}{10}\right) + (4)\left(\frac{5}{10}\right) = \frac{28}{10}$$

$$E(5X) = 5 * E(X) = 5 * \frac{28}{10} = 14$$

$$E(5X) = 5\, E(X) = 5\ \frac{28}{10} = 14$$

$$E(X^2)\ = \sum_{\text{all x}} x^2\, P(x) = (1)^2\left(\frac{2}{10}\right) + (2)^2\left(\frac{3}{10}\right) + (4)^2\left(\frac{5}{10}\right) = \frac{94}{10}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{94}{10} - \left(\frac{28}{10}\right)^2 = \frac{940 - 784}{100} = \frac{156}{100}$$

$$Var(2X) = 4 * Var(X) = 4 * \frac{156}{100} = \frac{624}{100}$$

$$Var(2X - 1) = 4 * Var(X) = \frac{624}{100}$$