Chapter Two

Probability

<u>Def. 1</u>: Probability

Probability is a numerical measure of the likelihood that an event will occur. Thus, probabilities can be used as measures of the degree of uncertainty associated with the events. If probabilities are available, we can determine the likelihood of each event occurring.

Probability values are always assigned on a scale from 0 to 1. A probability near zero indicates an event is unlikely to occur; a probability near 1 indicates an event is almost certain to occur. Other probabilities between 0 and 1 represent degree of likelihood that an event will occur.

<u>Def. 2</u>: Experiment

Defined as a process that generates well – defined outcomes.

<u>Def. 3</u>: Random Experiments

When different outcomes are obtained in repeated trials, the experiment is called a random experiment. Some sources of the variation in outcomes are controllable and some are not in the random experiment.

<u>Def. 4</u>: Sample Space

The sample space for an experiment is the set of all experimental outcomes, or is the set of all possible outcomes of a random experiment and is denoted as $\boldsymbol{S}$.

<u>Def. 5</u>: Event

An event is a subset of $\boldsymbol{S}$ (sample space) that means to indicate an outcome or collection of outcomes in any random experiment.

<u>Def. 6</u>: Mutually Exclusive Events

Events defined in such a way that the occurrence of one event precludes the occurrence of any of the other events. If one of the events happens the other cannot happen, are called mutually exclusive events.

Def. 7:

If a random experiment, whose sample space is $\boldsymbol{S}$, can result in n mutually exclusive and equally likely outcomes (i.e. each outcome has the same chance for occurrence) and if $n(A)$ of these outcomes have an attribute A, taken the probability of A, denoted by $P(A)$, is defined by:

$$P(A) = \frac{n(A)}{n} = \frac{number\ of\ outcomes\ of\ the\ event\ A}{total\ number\ of\ possible\ outcomes}$$

Ex.1: A fair coin is tossed once, then $S = \{H, T\}$, thus the probability of getting a head is $P(H) = \frac{1}{2}$ and the probability of getting a tail is $P(T) = \frac{1}{2}$.

Ex.2: When an unbiased die is thrown once, then $S = \{1,2,3,4,5,6\}$ the six outcomes are mutually exclusive because two or more faces cannot turn up simultaneously, then

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

To find the probability of getting an odd number, define $A = \{1,3,5\}$ then

$$P(A) = \frac{n(A)}{n} = \frac{3}{6} = \frac{1}{2}$$

Axioms of Probability

Let $\boldsymbol{S}$ be a sample space, and A is any event in the sample space $\boldsymbol{S}$, then $P(A)$ is called the probability of the event A if the following axioms hold:

1. For every event A, $P(A) \geq 0$.
2. $P(\boldsymbol{S}) = 1$.
3. If A and B are mutually exclusive events, then
   $P(A \cup B) = P(A) + P(B)$
4. If $A_1, A_2, \dots \dots$ are a sequence of mutually exclusive events, then
   $P(A_1 \cup A_2 \cup \dots \dots) = P(A_1) + P(A_2) + \cdots \cdots$
   Or $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Theorem 1: $P(\phi) = 0$ for any sample space S.

Proof:

Let A be any event, in the sample space $\boldsymbol{S}$

Then $A \cup \phi = \mathrm{A}$

$P(A \cup \phi) = P(\mathrm{A})$

$P(A) + \mathrm{P}(\phi) = P(\mathrm{A})$ axiom (3)

$P(\phi) = P(\mathrm{A}) - P(\mathrm{A}) = 0$

$\therefore P(\phi) = 0$

Theorem 2: For any finite sequence of disjoint events, $A_1, A_2, \dots\dots, A_n$, then $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$.

Proof:

Let $A_i \neq \phi \quad \forall\, i \leq n \qquad and\ \ A_i = \phi \quad \forall\, i > n$

Then $P(A_i) > 0 \qquad \forall\, i \leq n$

$P(A_i) = 0 \qquad \forall\, i > n$

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots\cdots \cup A_n \cup A_{n+1} \cup \cdots\cdots$$

$$= A_1 \cup A_2 \cup \cdots\cdots \cup A_n \cup \phi \cup \phi \cdots\cdots$$

$$= A_1 \cup A_2 \cup \cdots\cdots \cup A_n$$

$$= \bigcup_{i=1}^{n} A_i$$

$$\therefore P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

$$= \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(A_i)$$

$$= \sum_{i=1}^{n} P(A_i) + 0$$

Then $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

Theorem 3: For any event $A \subset S$, $P(A^c) = 1 - P(A)$.

Proof:

Since $A \ and \ A^c$ are disjoint events and $A \cup A^c = S$

Then $P(A \cup A^c) = P(S)$

And $P(A) + P(A^c) = P(S)$

Since $P(S) = 1$ axiom (2)

Then $P(A) + P(A^c) = 1$

$\therefore P(A^c) = 1 - P(A)$

Theorem 4: For any event A, $0 \le P(A) \le 1$.

Proof: a. $P(A) \ge 0$ $axiom\ (1)$

b. To prove $P(A) \le 1$

If not, Let $P(A) > 1$ and we know that

$P(A) + P(A^c) = 1$

Then $P(A^c) < 0$ $(contradiction)$

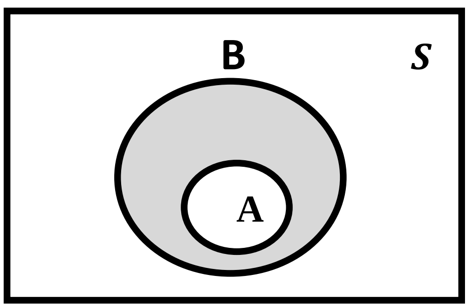
$\therefore P(A) \le 1$

From a and b, we have $0 \le P(A) \le 1$.

Theorem 5: If $A \subset B, then$ $P(A) \le P(B)$.

Proof:

From the figure we have



$A \subset B, \ and \ B = A \cup (BA^c)$

$P(B) = P(A \cup BA^c)$

$P(B) = P(A \cup BA^c) = P(A) + P(BA^c)$

$P(B) = P(A) + P(BA^c)$, $P(BA^c) \geq 0$

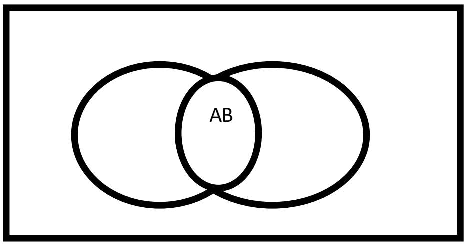
Then $P(B) \geq P(A)$

$\therefore$ $P(A) \leq P(B)$

Theorem 6: For any two events A and B

$P(A \cup B) = P(A) + P(B) - P(AB)$

Proof:



From the figure, we have

$A \cup B = AB^c \cup AB \cup A^cB$

Since AB^c, AB and A^cB are disjoint

Then $P(A \cup B) = P(AB^c) + P(AB) + P(A^cB)$

From the figure $P(AB^c) = P(A) - P(AB)$

$P(A^cB) = P(B) - P(AB)$

Then $P(A \cup B) = P(A) - P(AB) + P(AB) + P(B) - P(AB)$

$\therefore$ $P(A \cup B) = P(A) + P(B) - P(AB)$

Corollary 1: $P(A \cup B) \leq P(A) + P(B)$**.**

Corollary 2: For any three events A,B, and C

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

Corollary 3: $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$**.**

Ex.3: Two coins are tossed, what is the probability that at least one head appears?

Solution:

$S = \{HH, HT, TH, TT\}$

Let A be the event of at least are head

$A = \{HH, HT, TH\}$

$$P(A) = \frac{n(A)}{n} = \frac{3}{4}$$

Ex.4: A die is thrown once, define the events

A: an odd number appears.

B: number appearing is divisible.

Solution:

$S = \{1,2,3,4,5,6\}$, $A = \{1,3,5\}$, $B = \{3,6\}$

$$P(A) = \frac{n(A)}{n} = \frac{3}{6} = \frac{1}{2}$$

$$P(B) = \frac{n(B)}{n} = \frac{2}{6} = \frac{1}{3}$$

$$P(A^c B) = P(B) - P(AB)$$

$$AB = \{3\} \Rightarrow P(AB) = \frac{1}{6}$$

$$\therefore\ P(A^c B) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

Ex.5: Two die are thrown once, Let A and B be two events defined by:

A: the first die shows the number 1.

B: the sum of the two numbers appearing is less than 6.

Find $P(A)$, $P(B)$, $P(A \cup B)$, $P(A \cup B^c)$.

Solution:

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

$A = \{(1,1)(1,2)(1,3)(1,4)(1,5)(1,6)\}$

$B = \{(1,1)(1,2)(1,3)(1,4)(2,1)(2,2)(2,3)(3,1)(3,2)(4,1)\}$

$$\Rightarrow P(A) = \frac{n(A)}{n} = \frac{6}{36} = \frac{1}{6}$$

$$\Rightarrow P(B) = \frac{n(B)}{n} = \frac{10}{36} = \frac{5}{18}$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(AB)$$

AB: the first die shows the number (1) and the sum of two numbers appearing is less than (6).

$AB = \{(1,1)(1,2)(1,3)(1,4)\}$

$$P(AB) = \frac{n(AB)}{n} = \frac{4}{36}$$

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

$$P(A \cup B) = \frac{6}{36} + \frac{10}{36} - \frac{4}{36} = \frac{12}{36}$$

$$\Rightarrow P(A \cup B^c) = P(A) + P(B^c) - P(AB^c)$$

$$P(B^c) = 1 - P(B) = 1 - \frac{10}{36} = \frac{26}{36}$$

$$P(AB^c) = P(A) - P(AB)$$

$$P(AB^c) = \frac{6}{36} - \frac{4}{36} = \frac{2}{36}$$

$$P(A \cup B^c) = P(A) + P(B^c) - P(AB^c)$$

$$P(A \cup B^c) = \frac{6}{36} + \frac{26}{36} - \frac{2}{36} = \frac{30}{36}$$