<u>Def. 8</u>: Independence Events

- **Independence between two Events**: Events A and B are independent events if the probability of occurring together is the product of their individual probabilities.

$$P(A \cap B) = P(AB) = P(A) * P(B)$$

- **Dependence of two Events:** If events A and B are not independent, then they are dependent, which means that the occurrence of one event affects the probability of the other event.

$$P(A \cap B) = P(AB) \neq P(A) * P(B)$$

<u>Ex.9</u>: Suppose that a balanced die is rolled. Let A be the event that an even number is obtained, and Let B be the event that one of the numbers 1,2,3 or 4 is obtained. We shall show that the events A and B are independent.

Solution:

$$\Rightarrow S = \{1,2,3,4,5,6\}$$

$$A = \{2,4,6\} \quad \Rightarrow \quad P(A) = \frac{3}{6} = \frac{1}{2}$$

$$B = \{1,2,3,4\} \quad \Rightarrow \quad P(A) = \frac{4}{6} = \frac{2}{3}$$

Now, $A \cap B = A \text{ and } B$we obtained even number and obtained 1, 2, 3 or 4

$$\Rightarrow A \cap B = \{2,4\} \;\Rightarrow\; P(AB) = \frac{2}{6} = \frac{1}{3}$$

$$\Rightarrow P(AB) = P(A) * P(B)$$

$$\frac{1}{3} = \frac{1}{2} * \frac{2}{3}$$

$$\frac{1}{3} = \frac{1}{3}$$

<u>Theorem 7:</u> If two events A and B are independent, then the events A and B^c are also independent.

Proof:

We have $P(AB) = P(A) * P(B)$, and to show that

$P(AB^c) = P(A) * P(B^c)$

$AB^c = A - AB$

$P(AB^c) = P(A) - P(AB) \quad , \quad P(AB) = P(A) * P(B)$

$P(AB^c) = P(A) - P(A) * P(B)$

$P(AB^c) = P(A)[1 - P(B)]$

$P(AB^c) = P(A) * P(B^c)$

$\therefore A, B^c$ are independent events.

Theorem 8: If the two events A and B are independent, then the events $A^c$and B are also independent.

Proof:

We have $P(AB) = P(A) * P(B)$, and to show that

$P(A^cB) = P(A^c) * P(B)$

$A^cB = B - AB$

$P(A^cB) = P(B) - P(AB) \quad , \quad P(AB) = P(A) * P(B)$

$P(A^cB) = P(B) - P(A) * P(B)$

$P(A^cB) = P(B)[1 - P(A)]$

$P(A^cB) = P(B) * P(A^c)$

$\therefore A^c, B$ are independent events.

Theorem 9: If the two events A and B are independent, then the events $A^c$and B^c are also independent.

Proof:

We have $P(AB) = P(A) * P(B)$, and to show that

$P(A^c B^c) = P(A^c) * P(B^c)$

$A^c \cap B^c = (A \cup B)^c = 1 - (A \cup B)$

$P(A^c \cap B^c) = P(A^c B^c) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(AB)]$

$= 1 - P(A) - P(B) + P(AB)$

$P(A^c B^c) = 1 - P(A) - P(B) + P(A) * P(B)$

$= \left(1 - P(A)\right) * (1 - P(B))$

$P(A^c B^c) = P(A^c) * P(B^c)$

$\therefore A^c, B^c$ are independent events.

<u>Def. 9</u>: Independence of Three or More Events

The k events $A_1, A_2, \ldots A_k$ are independent if

$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1) * P(A_2) \cdots P(A_k)$.

In particular, in order for three events A,B and C to be independent, the following four relations must be satisfied:

1. $P(AB) = P(A) * P(B)$
2. $P(AC) = P(A) * P(C)$
3. $P(BC) = P(B) * P(C)$

And

4. $P(ABC) = P(A) * P(B) * P(C)$

<u>Def. 10</u>: Pairwise Independence of Events

The events A,B and C are said to be pairwise independence if the following relations are satisfied:

1. $P(AB) = P(A) * P(B)$
2. $P(AC) = P(A) * P(C)$
3. $P(BC) = P(B) * P(C)$

4. $P(ABC) \neq P(A) * P(B) * P(C)$

Ex.10: Two dice are thrown once, Let:

A: The sum of the two numbers is less than 5.

B: The first die shows the number 3.

C: The second die shows number 1 or 5 and the sum of the two numbers is less than 8 and more than 3.

Show that the events A,B and C are pairwise independent.

Solution:

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

$A = \{(1,1)(1,2)(1,3)(2,1)(2,2)(3,1)\}$

$B = \{(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)\}$

$C = \{(3,1)(4,1)(5,1)(6,1)(1,5)(2,5)\}$

$$\Rightarrow P(A) = \frac{n(A)}{n} = \frac{6}{36}$$

$$\Rightarrow P(B) = \frac{n(B)}{n} = \frac{6}{36}$$

$$\Rightarrow P(C) = \frac{n(C)}{n} = \frac{6}{36}$$

$$AB = \{(3,1)\} \;\Rightarrow\; P(AB) = \frac{1}{36}$$

$$AC = \{(3,1)\} \;\Rightarrow\; P(AC) = \frac{1}{36}$$

$$BC = \{(3,1)\} \;\Rightarrow\; P(BC) = \frac{1}{36}$$

$A \cap B \cap C = \{(3,1)\} \quad \Rightarrow \quad P(ABC) = \frac{1}{36}$

$\Rightarrow P(AB) = P(A) * P(B)$

$$\frac{1}{36} = \frac{1}{6} * \frac{1}{6} = \frac{1}{36}$$

$\Rightarrow P(AC) = P(A) * P(C)$

$$\frac{1}{36} = \frac{1}{6} * \frac{1}{6} = \frac{1}{36}$$

$\Rightarrow P(BC) = P(B) * P(C)$

$$\frac{1}{36} = \frac{1}{6} * \frac{1}{6} = \frac{1}{36}$$

$\Rightarrow P(ABC) = P(A) * P(B) * P(C)$

$$\frac{1}{36} \neq \frac{1}{6} * \frac{1}{6} * \frac{1}{6}$$

$\therefore P(ABC) \neq P(A) * P(B) * P(C)$

∴ A, B and C are pairwise independent events.