Conditional Probability

If A and B are two events of a probability space S, then the conditional probability of event A on the occurrence of event B, denoted by $P(A/B)$ is defined as:

$$P(A/B) = \frac{P(AB)}{P(B)} \quad , \quad P(B) > 0$$

This symbol may be read (probability of A given B) and the conditional probability of B given A is defined as:

$$P(B/A) = \frac{P(AB)}{P(A)} \quad , \quad P(A) > 0$$

Notes:

1. If A and B are dependent events, then

   a. $P(AB) = P(A/B) * P(B)$
   b. $P(AB) = P(B/A) * P(A)$

2. If A and B are independent events, then

   a. $P(A/B) = \frac{P(AB)}{P(B)} = \frac{P(A)*P(B)}{P(B)} = P(A)$
   b. $P(B/A) = \frac{P(AB)}{P(A)} = \frac{P(A)*P(B)}{P(A)} = P(B)$

Ex.11: Suppose that two dice were rolled and it was observed that the sum of the two numbers was odd, what is probability that the sum was less than 8?

Solution:

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

A: The sum of the two numbers was odd.

B: The sum of the two numbers was less than 8.

$$P(B/A) = \frac{P(AB)}{P(A)}$$

$$\Rightarrow A = \begin{Bmatrix} (1,2)(1,4)(1,6)(2,1)(2,3)(2,5)(3,2)(3,4)(3,6)(4,1) \\ (4,3)(4,5)(5,2)(5,4)(5,6)(6,1)(6,3)(6,5) \end{Bmatrix}$$

$$\Rightarrow P(A) = \frac{n(A)}{n} = \frac{18}{36}$$

$$\Rightarrow B = \begin{Bmatrix} (1,1)(1,2)(1,3)(1,4)(1,5)(1,6)(2,1)(2,2)(2,3)(2,4) \\ (2,5)(3,1)(3,2)(3,3)(3,4)(4,1)(4,2)(4,3)(5,1)(5,2)(6,1) \end{Bmatrix}$$

AB: The sum of the two numbers is odd and the sum was less than 8.

$$\Rightarrow AB = \begin{Bmatrix} (1,2)(1,4)(1,6)(2,1)(2,3)(2,5)(3,2)(3,4)(4,1)(4,3) \\ (5,2)(6,1) \end{Bmatrix}$$

$$\Rightarrow P(AB) = \frac{n(AB)}{n} = \frac{12}{36}$$

$$\Rightarrow P(B/A) = \frac{P(AB)}{P(A)} = \frac{12/36}{18/36} = \frac{12}{18} = \frac{2}{3}$$

Ex.12: A box contains black and white balls. Each ball is labeled either Y or Z. The composition of the box is shown below:

	Black	White	Total
Y	5	3	8
Z	1	2	3
Total	6	5	11

Let us now assume that a ball is to be selected at random from this box.

1. What is the probability of getting a black ball if it was labeled Y?
2. What is the probability of getting a black ball if it was labeled Z?

Solution:

A: The event that a randomly selected ball is labeled Y.

B: The event that a randomly selected ball is black.

C: The event that a randomly selected ball is labeled Z.

1. $P(A) = \frac{8}{11}$, $P(B) = \frac{6}{11}$, $P(AB) = \frac{5}{11}$

$$P(B/A) = \frac{P(AB)}{P(A)} = \frac{5/11}{8/11} = \frac{5}{8}$$

2. $(C) = \frac{3}{11}$, $P(B) = \frac{6}{11}$, $P(BC) = \frac{1}{11}$

$$P(B/C) = \frac{P(BC)}{P(C)} = \frac{1/11}{3/11} = \frac{1}{3}$$

<u>Ex.13</u>: Suppose there are (15) light bulbs in total, with (3) defective and 12 good ones, you start testing the bulbs one by one until you find all three defective bulbs. What is the probability that you will find the last defective bulb on the seventh test?

Solution:

Let E: The event of finding (2) defective among the first (6) tested.

F: The event of finding the third defective on the seventh testing.

So the number of ways to choose (2) defective bulbs out of (3) in the first six tests is given by the combination is C_2^3.

The number of ways to choose (4) good bulbs out of (12) in the first six tests is given by the combination is C_4^{12}.

$$\Rightarrow P(EF) = P(E) * P(F/E)$$

$$\Rightarrow P(E) = \frac{C_2^3 * C_4^{12}}{C_6^{15}} = \frac{3 * 495}{5005} = 0.296$$

$$\Rightarrow P(F/E) = \frac{1}{9} = 0.111$$

$$\Rightarrow P(EF) = P(E) * P(F/E)$$

$$= 0.296 * 0.111 = 0.032$$

<u>Def. 12</u>: Three Conditional events or more

If we have three events A, B and C, then

$$P(ABC) = P(A) * P(B/A) * P(C/AB)$$

In general, if we have (n) events, then

$$P(A_1A_2 \dots A_n) = P(A_1) * P(A_2/A_1) * P(A_3/A_1A_2) \dots P(A_n/A_1A_2 \dots \dots A_{n-1})$$

Ex.14: Four cards are to be drawn at random and without replacement from an ordinary deck of playing cards. What is the probability of receiving a spade, heart, diamond and a club?

Solution:

Let A: the event of receiving a spade.

B: the event of receiving a heart.

C: the event of receiving a diamond.

D: the event of receiving a club.

$$P(ABCD) = P(A) * P(B/A) * P(C/AB) * P(D/ABC)$$

$$= \frac{13}{52} * \frac{13}{51} * \frac{13}{50} * \frac{13}{49} = \frac{2197}{519800} = 0.0042$$

Ex.15: Let A and B be two events with, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(AB) = \frac{1}{4}$. Find $P(A/B), P(B/A), P(A \cup B), P(A^c/B^c), P(B^c/A^c)$.

Solution:

$$\Rightarrow P(A/B) = \frac{P(AB)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$\Rightarrow P(B/A) = \frac{P(AB)}{P(A)} = \frac{1/4}{1/2} = \frac{2}{4}$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(AB)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$$

$$\Rightarrow P(A^c/B^c) = \frac{P(A^cB^c)}{P(B^c)} = \frac{P(A \cup B)^c}{1 - P(B)} = \frac{1 - P(A \cup B)}{1 - P(B)}$$

$$= \frac{1 - \frac{7}{12}}{1 - \frac{1}{3}} = \frac{\frac{5}{12}}{\frac{2}{3}} = \frac{5}{12} * \frac{3}{2} = \frac{5}{8}$$

$$\Rightarrow P(B^c/A^c) = \frac{P(A^cB^c)}{P(A^c)} = \frac{P(A \cup B)^c}{1 - P(A)} = \frac{1 - P(A \cup B)}{1 - P(A)}$$

$$= \frac{1 - \frac{7}{12}}{1 - \frac{1}{2}} = \frac{\frac{5}{12}}{\frac{1}{2}} = \frac{5}{12} * \frac{2}{1} = \frac{5}{6}$$

<u>Ex.16</u>: A student will get passing grades in Mathematics and English, or in both with probabilities are respectively, $P(M) = 0.70$, $P(E) = 0.80$ and $P(ME) = 0.56$. Are the events M and E independent?

Solution:

$$\Rightarrow P(M/E) = \frac{P(ME)}{P(E)} = \frac{0.56}{0.80} = 0.70$$

And since $P(M) = 0.70$, therefore $P(M/E) = P(M) = 0.70$ and we find that M and E are independent.