Def. 13: **Bayes Law**

Theorem 10: General theorem for conditional probabilities

Suppose that $A_1, A_2, \dots, A_m$ are mutually exclusive events, such that

$A_1 \cup A_2 \cup \dots \cup A_m = S \quad and \;\; P(A_i) > 0 \quad where\ i = 1,2,\dots,m$

Then for any event B, we have:

$$P(B) = P(B/A_1)P(A_1) + P(B/A_2)P(A_2) + \cdots + P(B/A_m)P(A_m)$$

Proof:

$B = B \cap S$

$\quad = B \cap (A_1 \cup A_2 \cup \dots \cup A_m)$

$\quad = (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_m)$

$\quad = (BA_1) \cup (BA_2) \cup \dots \cup (BA_m)$

We know that

$$P(B/A) = \frac{P(AB)}{P(A)} \;\Rightarrow\; P(AB) = P(B/A) * P(A)$$

$\Rightarrow P(B) = P(BA_1) + P(BA_2) + \cdots + P(BA_m)$

$\therefore P(B) = P(B/A_1)P(A_1) + P(B/A_2)P(A_2) + \cdots + P(B/A_m)P(A_m)$

Ex.17: Consider two urns, urn A_1 contains (5) white balls and (7) red balls, urn A_2 contains (6) white balls and (4) red balls. One of the urns is selected at random and a ball is drawn from it. Find the probability that the ball drawn will be white.

Solution:

Let W: the event that white ball is drawn.

$P(W) = P(WA_1) + P(WA_2)$

$P(W) = P(W/A_1) * P(A_1) + P(W/A_2) * P(A_2)$

$\Rightarrow P(A_1) = P(A_2) = \frac{1}{2}$

$$\Rightarrow P(W/A_1) = \frac{C_1^5}{C_1^{12}} = \frac{5}{12}$$

$$\Rightarrow P(W/A_2) = \frac{C_1^6}{C_1^{10}} = \frac{3}{5}$$

$$P(W) = \frac{5}{12} * \frac{1}{2} + \frac{3}{5} * \frac{1}{2} = 0.51$$

<u>Theorem 11:</u> Bayes Theorem

Let the events $A_1, A_2, \dots, A_k$ form a partition of the space S such that $P(A_j) > 0 \quad \forall\, j = 1,2,\dots,k$, and let B be any event in S such that $P(B) > 0$. Then $\forall\, i = 1,2,\dots,k$

$$P(A_i/B) = \frac{P(A_i) * P(B/A_i)}{\sum_{j=1}^{k} P(A_j) * P(B/A_j)}$$

$$P(B) = \sum_{j=1}^{k} P(A_j) * P(B/A_j)$$

$$P(A_i B) = P(A_i) * P(B/A_i)$$

<u>Ex.18</u>: Three different machines M_1, M_2, and M_3 are producing similar items. Suppose that 20% of the items were produced by machine M_1, 30% by machine M_2, and 50% by machine M_3. Suppose that 1% of the items produced by machine M_1 are defective, 2% of the items produced by machine M_2 are defective, 3% of the items produced by machine M_3 are defective. Finally suppose that one item is selected at random from the entire batch.

a. What is the probability that this item is defective?
b. If the selected item is found to be defective, What is the probability that this item was produced by machine M_2?

Solution:

Let D: the event that an item is defective.

M_1: the event that an item was produced by machine M_1.

M_2: the event that an item was produced by machine M_2.

M_3: the event that an item was produced by machine M_3.

We are given:

$\Rightarrow P(M_1) = 0.20\,, P(M_2) = 0.30\,, P(M_3) = 0.50$

$\Rightarrow P(D/M_1) = 0.01\,, P(D/M_2) = 0.02\,, P(D/M_3) = 0.03$

a. $P(D) = P(DM_1) + P(DM_2) + P(DM_3)$

$P(D) = P(D/M_1) * P(M_1) + P(D/M_2) * P(M_2) + P(D/M_3) * P(M_3)$

$P(D) = 0.01 * 0.20 + 0.02 * 0.30 + 0.03 * 0.50 = 0.023$

So, the probability that an item is defective is 0.023 or 2.3%.

b. We need to find $P(M_2/D)$, the probability that an item was produced by machine M_2 given that it is defective. Using Bayes theorem:

$$P(M_2/D) = \frac{P(M_2) * P(D/M_2)}{\sum_{j=1}^{3} P(M_j) * P(D/M_j)} = \frac{0.30 * 0.02}{0.023} = \frac{0.006}{0.023} = 0.2609$$

So, the probability that a defective item was produced by machine M_2 is approximately 0.26090 or 26.09%.

<u>Ex.19</u>: In a certain collage 4% of the men and 1% of the women are taller than 180 cm. furthermore, 60% of the students are women. Now, if a student is selected at random:

a. What is the probability that the student is taller than 180 cm?
b. Given that the student is taller than 180 cm, what is the probability that the student is a woman?

Solution:

Let T: the event that a student is taller than 180 cm.

M: the event that a student is a man.

W: the event that a student is a woman.

Hence:

The probability that a man is taller than 180 cm, $P(T/M) = 0.04$.

The probability that a woman is taller than 180 cm, $P(T/W) = 0.01$.

The probability that a student is a woman, $P(W) = 0.60$.

The probability that a student is a man, $P(M) = 0.40$, [Since $P(M) = 1 - P(W)$].

a. We need to find $P(T)$, the overall probability that a randomly selected student is taller than 180 cm. we use the law of total probability:
   $P(T) = P(TW) + P(TM)$
   $P(T) = P(T/W) * P(W) + P(T/M) * P(M)$
   $P(T) = 0.01 * 0.60 + 0.04 * 0.40 = 0.022$

So, the probability that a randomly selected student is taller than 180 cm is 0.022 or 2.2%.

b. We need to find $P(W/T)$, the probability that the student is a woman given that they are taller than 180 cm.

$$P(W/T) = \frac{P(W) * P(T/W)}{P(T/W) * P(W) + P(T/M) * P(M)} = \frac{0.60 * 0.01}{0.022}$$
$$= 0.2727$$

So, the probability that a student who is taller than 180 cm is a woman is approximately 0.2727 or 27.27%.

<u>Ex.20</u>: Urn A_1 contain (8) black and (2) white marbles. Urn A_2 contain (3) black and (7) white marbles, and Urn A_3 contain (5) black and (5) white marbles.

A fair die is tossed. If the die shows 1,2, or 3, a marble is selected from Urn A_1. If the die shows 4 or 5, a marble is selected from Urn A_2. If the die shows 6, a marble is selected from Urn A_3.

a. What is the probability that the marble drawn is black?
b. If the marble drawn is black, What is the probability that it was chosen from Urn A_2.

Solution:

Probability of choosing Urn $A_1 \Rightarrow P(A_1) = \frac{3}{6} = \frac{1}{2}$

Probability of choosing Urn $A_2 \Rightarrow P(A_2) = \frac{2}{6} = \frac{1}{3}$

Probability of choosing Urn $A_3 \Rightarrow P(A_3) = \frac{1}{6}$

Probability of drawing a black marble from Urn A_1:

$$P(B/A_1) = \frac{C_1^8}{C_1^{10}} = \frac{8}{10} = \frac{4}{5}$$

Probability of drawing a black marble from Urn A_2:

$$P(B/A_2) = \frac{C_1^3}{C_1^{10}} = \frac{3}{10}$$

Probability of drawing a black marble from Urn A_3:

$$P(B/A_3) = \frac{C_1^5}{C_1^{10}} = \frac{5}{10} = \frac{1}{2}$$

a. $P(B) = P(BA_1) + P(BA_2) + P(BA_3)$
$P(B) = P(B/A_1) * P(A_1) + P(B/A_2) * P(A_2) + P(B/A_3) * P(A_3)$
$P(B) = \frac{4}{5} * \frac{1}{2} + \frac{3}{10} * \frac{1}{3} + \frac{1}{2} * \frac{1}{6} = 0.5833$

So, the probability of drawing a black marble is approximately 0.5833 or 58.33%.

b. $P(A_2/B) = \frac{P(A_2)*P(B/A_2)}{P(B)} = \frac{P(A_2)*P(B/A_2)}{\sum_{j=1}^{3} P(A_j)*P(B/A_j)} = \frac{\frac{2}{6}*\frac{3}{10}}{0.5833} = 0.1714$

So, the probability that the black marble was chosen from Urn A_2 is approximately 0.1714 or 17.14%.

Problems

1. Let the sample space $S = \{x/0 < x < 1\}$, and $A = \left\{x/0 < x < \frac{1}{2}\right\}, B = \left\{x/\frac{1}{2} \leq x < 1\right\}$. Find $P(B)$ if $P(A) = \frac{1}{4}$.
2. If the sample space $S = A \cup B, P(A) = 0.8$ and $P(B) = 0.5$. Find $P(AB)$.
3. Let A, B and C be three mutually exclusive subsets of the sample space S. Find $P[(A \cup B) \cap C]$ and $P(A^c \cup B^c)$.
4. Let the subsets $A = \left\{x/\frac{1}{4} < x < \frac{1}{2}\right\}, and\ B = \left\{x/\frac{1}{2} \leq x < 1\right\}$, of the sample space $S = \{x/0 < x < 1\}$ be such that $P(A) = \frac{1}{8}\ and\ P(B) = \frac{1}{2}$. Find $P(AB), P(A^c) and\ P(A^c B^c)$.
5. The probability of the mutually exclusive events A and B are related as $P(B) = (P(A))^2$ and $A \cup B = S$ the sample space. Find $P(A)$.
6. Three light bulbs are chosen at random from 15 bulbs of which 5 are defective, find the probability that

   A: none is defective.

   B: Exactly one is defective.

   C: At least one is defective.

7. Three cards are drawn at random from a deck of 52 cards. Let the events A,B,C,D and E be defined as:

   A: an ace.

   B: 2 cards of diamond.

   C: 2 picture cards and one numbered card of spade.

   D: one numbered card, one Jack and the queen of hearts.

   E: the first card is an ace and the others one picture cards.

   Find the probability of the above events.

8. Let A and B be two events with $P(A \cup B) = \frac{3}{4}, P(A^c) = \frac{2}{3}\ and\ P(A \cap B) = \frac{1}{4}$. Find $P(A), P(B), P(A \cap B^c)$.
9. Let A and B are independent events, then prove that, $P(A/B) = P(A)\ and\ P(B/A) = P(B)$.

10. A pair of dice are thrown once. Let A, B and C be three events defined as:

A: the first die shows number 6.
B: the sum of the two dice is 7.
C: the second die shows odd number and the sum of the two dice is 7.

Find:

a. $P(A), P(B), P(C)$, $P(AB), P(A \cup B), P(AC), P(BC)$,

$P(A \cup C)$, $P(B \cup C)$

b. Are A and B independent?
C. Are A and C independent?

11. In a sample space S, three events A, B and C have the probabilities $P(A) = P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$, $P(AB) = \frac{1}{6}$, $P(AC) = \frac{1}{8}$, $P(BC) = 0$

a. Find $P(A \cup B \cup C)$
b. Are A and B independent?

12. In a factory, machine A produced 40% of the output and machine B produces 60% on the average 10% produces by A are defective and 5% produces by B are defective. What is the probability that it was produced by machine A?

13. Consider three boxes, A, B and C. Box A contains 4 red balls and 5 blue balls. Box B contains 5 red balls and 6 white balls. Box C contains 3 red balls and 4 black balls. If we select a box at random and draw a ball from the box.

a. What is the probability that the ball is red?
b. If the drawn ball is red, what is the probability that it came from box B?

14. Box A contains 9 cards numbered 1 through 9, and box B contains 5 cards numbered 1 through 5. A box is chosen at random and a card drawn from it,

a. What is the probability that the card shows an odd number?
b. If the drawn card shows an odd number, what is the probability that it came from box A?