Chapter Three

Random variables and their probability functions

Def. 1: **Random Variable**

A random variable X is a real valued function whose numerical value is determined by the outcome of a random experiment.

Ex.1: Two dice are thrown once. Define a random variable x to represent the sum of the two numbers shown by the two dice. Therefore, the random variable x can take on the values 2,3,4,…,12, which can be described as follows:

Solution:

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

$x = 2 \Rightarrow \{(1,1)\}$

$x = 3 \Rightarrow \{(1,2)\ (2,1)\}$

$x = 4 \Rightarrow \{(1,3)\ (3,1)(2,2)\}$

$x = 5 \Rightarrow \{(1,4)\ (4,1)(2,3)(3,2)\}$

$x = 6 \Rightarrow \{(1,5)\ (5,1)(3,3)(2,4)(4,2)\}$

$\vdots$

$x = 12 \Rightarrow \{(6,6)\}$

<u>Ex.2</u>: Let the random variable X be the number of white balls in a sample of size 2 draw without replacement from an urn containing 6 balls of which 4 are white in which the white balls have been numbered 1 to 4 and the remaining 2 balls 5 and 6. The random variable X takes on the values 0, 1, 2.

Solution:

$x = 0 \Rightarrow \{(5,6)\ (6,5)\}$

$$x = 1 \Rightarrow \begin{Bmatrix} (1,5)\ (1,6)(2,5)(2,6)\ (3,5)(3,6)(4,5)\ (4,6) \\ (5,1)(6,1)\ (5,2)(6,2)(5,3)\ (6,3)(5,4)(6,4) \end{Bmatrix}$$

$$x = 2 \Rightarrow \begin{Bmatrix} (1,2)\ (1,3)(1,4)(2,3)\ (2,4)(3,4) \\ (2,1)(3,1)\ (4,1)(3,2)(4,2)\ (4,3) \end{Bmatrix}$$

<u>Def. 2</u>: Discrete Random Variable

A random variable X that takes on distinct values only is called a discrete random variable.

<u>Ex.3</u>: Suppose that our experiment is tossing a fair coin. Let X denote the number of heads in the experiment. The sample space in this case is {head (H), tail (T)}. Then the variable takes on either 0 or 1. Thus X is a discrete random variable with,

Solution:

$x = 0 \Rightarrow \{T\}$

$x = 1 \Rightarrow \{H\}$

<u>Ex.4</u>: Now consider the roll of a single fair die. Let X denote the number of spots showing. The sample space is given by {1, 2, 3, 4, 5, 6}. Then X is a discrete random variable taking on one of the values 1, 2, 3, 4, 5, 6. The events corresponding to the values of X are

Solution:

$x = 1 \Rightarrow \{one\ spot\}$

$x = 2 \Rightarrow \{two\ spots\}$

$x = 3 \Rightarrow \{three\ spots\}$

$x = 4 \Rightarrow \{four\ spots\}$

$x = 5 \Rightarrow \{five\ spots\}$

$x = 6 \Rightarrow \{six\ spots\}$

<u>Def. 3</u>: Probability Mass Function

Let X be a discrete random variable and let $x_1, x_2, \dots\dots$ be distinct values that X may assume. Then the function $P(x)$ defined by:

$$P(x) = \begin{cases} P(X = x_i) & , X = x_i \quad , i = 1, 2, \dots\dots \\ 0 & , X \neq x_i \end{cases}$$

is defined to be the probability mass function of the random variable X, $P(x_i)$ is called the probability mass at value x_i.

Properties of the p.m.f.:

1. $0 \leq P(x_i) \leq 1$
2. $\sum_{i=1}^{\infty} P(x_i) = 1$

<u>Ex.5</u>: Three coins are thrown once. Let X be the number of heads obtained, then the values taken by x are 0, 1, 2, 3. Find the probability mass function of X.

Solution:

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$x = 0 \Rightarrow \{TTT\} \Rightarrow P(x = 0) = \frac{1}{8}$

$x = 1 \Rightarrow \{TTH, THT, HTT\} \Rightarrow P(x = 1) = \frac{3}{8}$

$x = 2 \Rightarrow \{HHT, HTH, THH\} \Rightarrow P(x = 2) = \frac{3}{8}$

$x = 3 \Rightarrow \{HHH\} \Rightarrow P(x = 3) = \frac{1}{8}$

$$\Rightarrow P(x) = \begin{cases} 1/8 & x = 0 \\ 3/8 & x = 1 \\ 3/8 & x = 2 \\ 1/8 & x = 3 \\ 0 & other\ wise \end{cases}$$

1. $0 \leq P(x_i) \leq 1$

$\Rightarrow \quad 0 \leq P(x_1), P(x_2), P(x_3), P(x_4) \leq 1$

2. $\sum_{i=1}^{\infty} P(x_i) = 1$

$$\Rightarrow \quad \sum_{i=0}^{3} P(x_i) = P(0) + P(1) + P(2) + P(3)$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$

Ex.6: Suppose that we throw one die. Let X be the number shown by this die. Find the probability mass function of this experiment.

Solution:

$S = \{1,2,3,4,5,6\}$

$x = 1 \ \Rightarrow \ \{1\} \ \Rightarrow \ P(x = 1) = \frac{1}{6}$

$x = 2 \ \Rightarrow \ \{2\} \ \Rightarrow \ P(x = 2) = \frac{1}{6}$

$x = 3 \ \Rightarrow \ \{3\} \ \Rightarrow \ P(x = 3) = \frac{1}{6}$

$x = 4 \ \Rightarrow \ \{4\} \ \Rightarrow \ P(x = 4) = \frac{1}{6}$

$x = 5 \ \Rightarrow \ \{5\} \ \Rightarrow \ P(x = 5) = \frac{1}{6}$

$x = 6 \ \Rightarrow \ \{6\} \ \Rightarrow \ P(x = 6) = \frac{1}{6}$

$$\Rightarrow P(x) = \begin{cases} 1/6 & x = 1 \\ 1/6 & x = 2 \\ 1/6 & x = 3 \\ 1/6 & x = 4 \\ 1/6 & x = 5 \\ 1/6 & x = 6 \\ 0 & o.w. \end{cases} \qquad \text{Or} \quad P(x) = \begin{cases} 1/6 & x = 1,2,3,4,5,6 \\ 0 & o.w. \end{cases}$$

1. $0 \leq P(x_i) \leq 1$

$\Rightarrow \ 0 \leq P(x_1), P(x_2), P(x_3), P(x_4), P(x_5), P(x_6) \leq 1$

2. $\sum_{i=1}^{\infty} P(x_i) = 1$

$$\Rightarrow \sum_{i=1}^{6} P(x_i) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

<u>Ex.7</u>: Two dice are thrown once. Define a r.v. X to be the sum of the two numbers shown by the two dice. Find the p.m.f. of X, we get the sample space as follows:

Solution:

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

$$x = 2 \ \Rightarrow \ \{(1,1)\} \ \Rightarrow \ P(x = 2) = \frac{1}{36}$$

$$x = 3 \ \Rightarrow \ \{(1,2)(2,1)\} \ \Rightarrow \ P(x = 3) = \frac{2}{36}$$

$$x = 4 \ \Rightarrow \ \{(1,3)(3,1)(2,2)\} \ \Rightarrow \ P(x = 4) = \frac{3}{36}$$

$$x = 5 \ \Rightarrow \ \{(1,4)(4,1)(2,3)(3,2)\} \ \Rightarrow \ P(x = 5) = \frac{4}{36}$$

$$x = 6 \;\Rightarrow\; \{(1,5)(5,1)(4,2)(2,4)(3,3)\} \;\Rightarrow\; P(x=6) = \frac{5}{36}$$

$$x = 7 \;\Rightarrow\; \{(1,6)(6,1)(4,3)(3,4)(5,2)(2,5)\} \;\Rightarrow\; P(x=7) = \frac{6}{36}$$

$$x = 8 \;\Rightarrow\; \{(2,6)(6,2)(3,5)(5,3)(4,4)\} \;\Rightarrow\; P(x=8) = \frac{5}{36}$$

$$x = 9 \;\Rightarrow\; \{(3,6)(6,3)(4,5)(5,4)\} \;\Rightarrow\; P(x=9) = \frac{4}{36}$$

$$x = 10 \;\Rightarrow\; \{(4,6)(6,4)(5,5)\} \;\Rightarrow\; P(x=10) = \frac{3}{36}$$

$$x = 11 \;\Rightarrow\; \{(5,6)(6,5)\} \;\Rightarrow\; P(x=11) = \frac{2}{36}$$

$$x = 12 \;\Rightarrow\; \{(6,6)\} \;\Rightarrow\; P(x=12) = \frac{1}{36}$$

$$\Rightarrow P(x) = \begin{cases} 1/36 & x = 2 \\ 2/36 & x = 3 \\ 3/36 & x = 4 \\ 4/36 & x = 5 \\ 5/36 & x = 6 \\ 6/36 & x = 7 \\ 5/36 & x = 8 \\ 4/36 & x = 9 \\ 3/36 & x = 10 \\ 2/36 & x = 11 \\ 1/36 & x = 12 \\ 0 & o.w \end{cases} \qquad \text{Or} \quad P(x) = \begin{cases} 1/36 & x = 2,12 \\ 2/36 & x = 3,11 \\ 3/36 & x = 4,10 \\ 4/36 & x = 5,9 \\ 5/36 & x = 6,8 \\ 6/36 & x = 7 \\ 0 & o.w. \end{cases}$$

$$\Rightarrow \quad 0 \le P(x_i) \le 1 \quad \forall\, x$$

$$\Rightarrow \quad \sum_{i=2}^{12} P(x_i) = 1$$

<u>Ex.8</u>: Determine the constant C so that $P(x)$ satisfies the condition of being a p.m.f. of x.

$$P(x) = \begin{cases} C(x+1)^2 & x = 0,1,2,3 \\ 0 & o.w. \end{cases}$$

Solution:

$$\Rightarrow \sum_{i=1}^{\infty} P(x_i) = 1$$

$$\Rightarrow C\sum_{i=0}^{3}(x+1)^2 = 1$$

$$\Rightarrow C[1^2 + 2^2 + 3^2 + 4^2] = 1$$

$$\Rightarrow C[30] = 1 \quad \Rightarrow \quad C = \frac{1}{30}$$

$$P(x) = \begin{cases} \frac{1}{30}(x+1)^2 & x = 0,1,2,3 \\ 0 & o.w. \end{cases} \quad \text{Or} \quad P(x) = \begin{cases} 1/30 & x = 0 \\ 4/30 & x = 1 \\ 9/30 & x = 2 \\ 16/30 & x = 3 \\ 0 & o.w. \end{cases}$$

<u>Ex.9</u>: An urn contains 4 balls numbered 1, 2, 3, 4 respectively, let X be the number that occurs if one ball is drawn at random from the urn. Write the p.m.f. of X, find $P(1 < x \leq 4), P(x \geq 2)$.

Solution:

$$\Rightarrow P(x) = \begin{cases} \frac{1}{4} & x = 1,2,3,4 \\ 0 & o.w. \end{cases} \quad \text{Or} \quad P(x) = \begin{cases} 1/4 & x = 1 \\ 1/4 & x = 2 \\ 1/4 & x = 3 \\ 1/4 & x = 4 \\ 0 & o.w. \end{cases}$$

$$\Rightarrow P(1 < x \leq 4) = \sum_{i=2}^{4} P(x) = P(x=2) + P(x=3) + P(x=4)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow P(x \geq 2) = \sum_{i=2}^{4} P(x) = P(x = 2) + P(x = 3) + P(x = 4)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

<u>Ex.10</u>: Verify the following functions are probability mass function:

a) $P(x) = \left(\frac{1}{2}\right)^x \qquad x = 1,2,3, \ldots.$

b) $P(x) = \left(\frac{1}{4}\right)^x \qquad x = 1,2,3, \ldots.$

Solution:

a)

$$\sum_{i=1}^{\infty} P(x_i) = 1$$

$$\Rightarrow \sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \ldots$$

$$= \frac{1/2}{1 - 1/2} = 1$$

$$\because \sum_{x=1}^{\infty} P(x) = 1$$

$\therefore$ $P(x)$ is p.m.f.

b)

$$\Rightarrow \sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = \left(\frac{1}{4}\right)^1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots \ldots = \frac{1/4}{1 - 1/4} = \frac{1/4}{3/4} = \frac{1}{3}$$

$$\because \sum_{x=1}^{\infty} P(x) \neq 1$$

$\therefore$ $P(x)$ is not p.m.f.

Def. 4: **Cumulative Distribution Function**

The cumulative probability function of a discrete random variable is denoted by $F(x)$ and is defined by:

$$F(x) = P(X \leq x) = \sum_{X_i \leq x} P(x_i)$$

We may refer sometime to the cumulative probability function by the cumulative distribution function (c.d.f.) or the Distribution Function (DF).

Properties of the c.d.f.:

1. $\lim_{x \to \infty} F(x) = 1$
2. $\lim_{x \to -\infty} F(x) = 0$
3. $F(x_1) \leq F(x_2) \quad if\ x_1 < x_2$
4. $P(a < x \leq b) = F(b) - F(a)$

Ex.11: For the following p.m.f., find the cumulative distribution function and draw it.

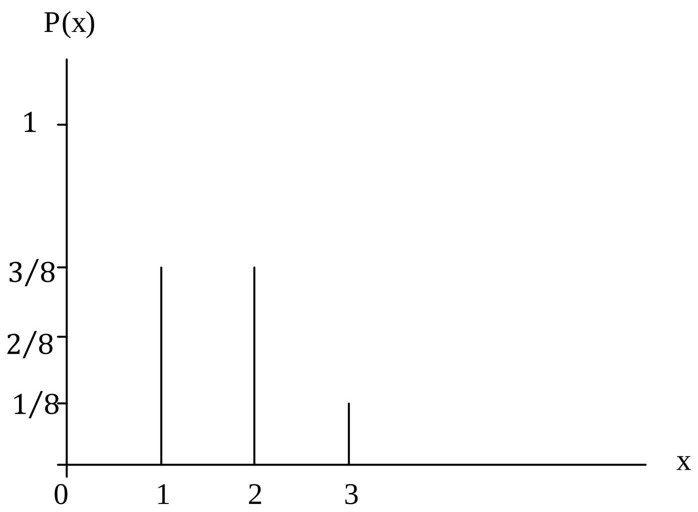
$$P(x) = \begin{cases} 1/8 & x = 0 \\ 3/8 & x = 1 \\ 3/8 & x = 2 \\ 1/8 & x = 3 \\ 0 & other\ wise \end{cases}$$

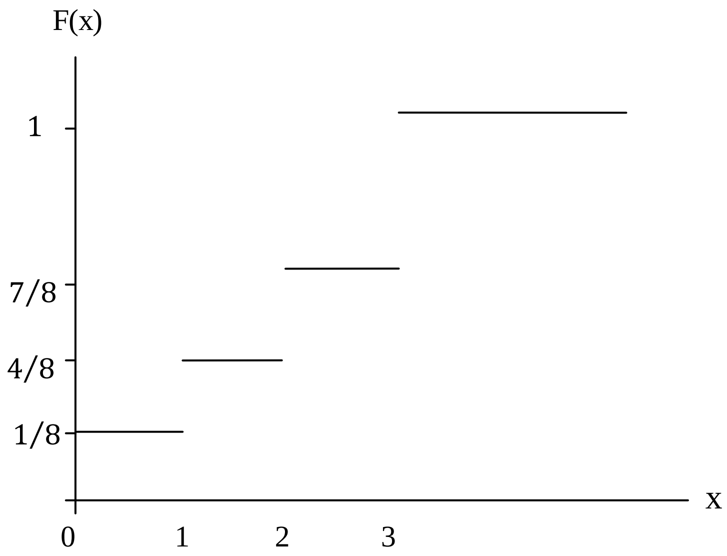
Solution:

$$\Rightarrow F(x) = P(X \leq x) = \sum_{X_i \leq x} P(x_i)$$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ 0 + 1/8 = 1/8 & 0 \leq x < 1 \\ 1/8 + 3/8 = 4/8 & 1 \leq x < 2 \\ 4/8 + 3/8 = 7/8 & 2 \leq x < 3 \\ 7/8 + 1/8 = 8/8 & x \geq 0 \end{cases}$$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ 1/8 & 0 \leq x < 1 \\ 4/8 & 1 \leq x < 2 \\ 7/8 & 2 \leq x < 3 \\ 8/8 & x \geq 0 \end{cases}$$





<u>Ex.12</u>: For the following function

$$P(x) = \begin{cases} x/k & x = 1,2,3,4,5 \\ 0 & o.w. \end{cases}$$

1. Find the value of k so that $P(x)$ satisfies the condition of being a probability mass function for a random variable x.
2. Find the c.d.f. of x.

Solution:

1. $P(x) = \begin{cases} x/k & x = 1,2,3,4,5 \\ 0 & o.w. \end{cases}$

$$\sum_{all\ x} P(x) = 1 \Rightarrow \sum_{x=1}^{5} \frac{x}{k} = 1$$

$$\Rightarrow \frac{1}{k}\sum_{x=1}^{5} x = 1$$

$$\Rightarrow \frac{1}{k}[1 + 2 + 3 + 4 + 5] = 1$$

$$\Rightarrow \frac{1}{k}[15] = 1 \quad \Rightarrow k = 15$$

$$\Rightarrow P(x) = \begin{cases} x/15 & x = 1,2,3,4,5 \\ 0 & o.w. \end{cases}$$

2. $P(x) = \begin{cases} 1/15 & x = 1 \\ 2/15 & x = 2 \\ 3/15 & x = 3 \\ 4/15 & x = 4 \\ 5/15 & x = 5 \\ 0 & o.w. \end{cases}$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/15 & 1 \le x < 2 \\ 3/15 & 2 \le x < 3 \\ 6/15 & 3 \le x < 4 \\ 10/15 & 4 \le x < 5 \\ 15/15 & 5 \le x \end{cases}$$

Ex.13: Let X be a discrete random variable, if the cumulative distribution function of X given by:

$$F(x)=\begin{cases} 0 & x<-2 \\ 2/8 & -2\le x<1 \\ 3/8 & 1\le x<2 \\ 7/8 & 2\le x<4 \\ 1 & x\ge 4 \end{cases}$$

1. Find p.m.f. of x.
2. Find $P(-2<x<4)$?
3. Find $P(x-5\ge -1)$?
4. Draw c.d.f. and p.m.f.

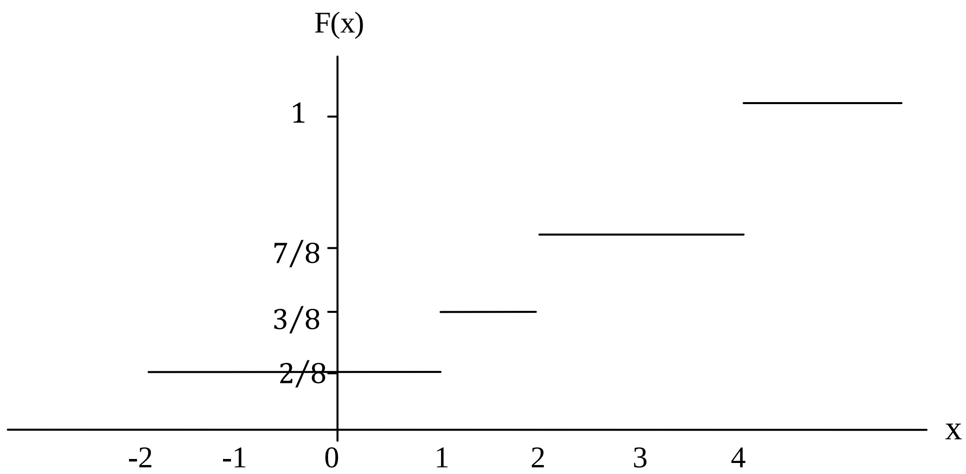
Solution:

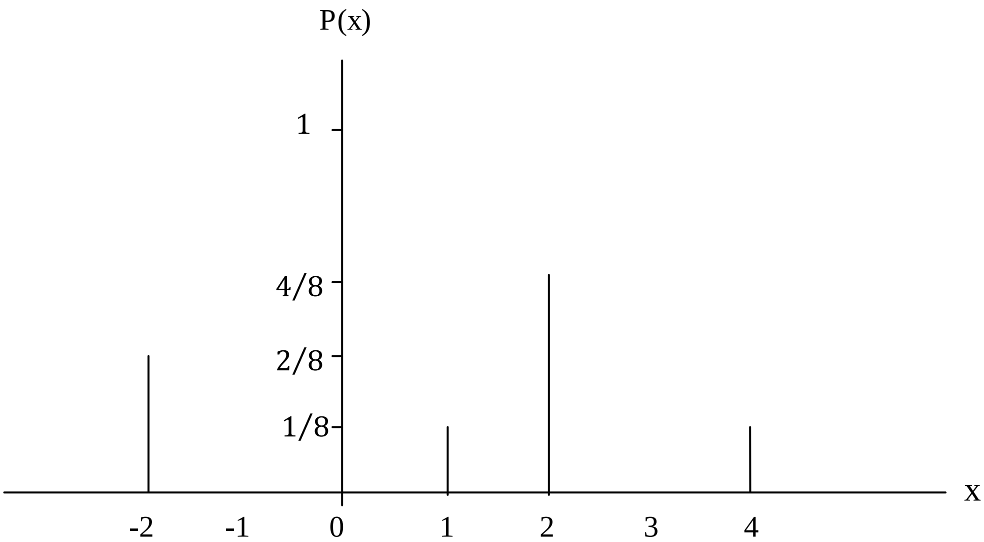
1. $$P(x)=\begin{cases} 2/8 & x=-2 \\ 3/8-2/8=1/8 & x=1 \\ 7/8-3/8=4/8 & x=2 \\ 8/8-7/8=1/8 & x=4 \\ 0 & o.w. \end{cases}$$

$$P(x)=\begin{cases} 2/8 & x=-2 \\ 1/8 & x=1 \\ 4/8 & x=2 \\ 1/8 & x=4 \\ 0 & o.w. \end{cases}$$

2. $P(-2<x<4)=P(x=1)+P(x=2)=\frac{1}{8}+\frac{4}{8}=\frac{5}{8}$

3. $P(x-5\ge -1)=P(x\ge 4)=\frac{1}{8}$





<u>Ex.14</u>: Two dice are thrown once. Let X be the absolute difference between the two numbers shown by the dice. Find the probability mass function of X and cumulative distribution function of X

Solution:

For this situation the random variable X takes on the values 0, 1, 2, 3, 4, and 5. And then the p.m.f. of X will be as:

$$\Rightarrow P(x) = \begin{cases} 6/36 & x = 0 \\ 10/36 & x = 1 \\ 8/36 & x = 2 \\ 6/36 & x = 3 \\ 4/36 & x = 4 \\ 2/36 & x = 5 \\ 0 & o.w. \end{cases}$$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ 6/36 & 0 \le x < 1 \\ 16/36 & 1 \le x < 2 \\ 24/36 & 2 \le x < 3 \\ 30/36 & 3 \le x < 4 \\ 34/36 & 4 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

<u>Ex.15</u>: Let X be a r.v. having a p.m.f.:

X	-2	-1	0	2	3	5
$P(x)$	C	2C	3C	2C	C	4C

1. Find the value of C.
2. Find the c.d.f.
3. Find $P(x > -2), P(x\ at\ least\ 2), P(x\ more\ than\ 2), P(-1 < x \leq 3)$.

Solution:

1. $\sum_{all\ x} P(x) = 1$

$$C + 2C + 3C + 2C + C + 4C = 1$$

$$13C = 1$$

$$\therefore\ C = \frac{1}{13}$$

2. $$P(x) = \begin{cases} 1/13 & x = -2 \\ 2/13 & x = -1 \\ 3/13 & x = 0 \\ 2/13 & x = 2 \\ 1/13 & x = 3 \\ 4/13 & x = 5 \\ 0 & o.w. \end{cases} \qquad F(x) = \begin{cases} 0 & x < -2 \\ 1/13 & -2 \leq x < -1 \\ 3/13 & -1 \leq x < 0 \\ 6/13 & 0 \leq x < 2 \\ 8/13 & 2 \leq x < 3 \\ 9/13 & 3 \leq x < 5 \\ 1 & x \geq 5 \end{cases}$$

3.

$\Rightarrow P(x > -2)$

$$= P(x = -1) + P(x = 0) + P(x = 2) + P(x = 3) + P(x = 5)$$

$$= \frac{2}{13} + \frac{3}{13} + \frac{2}{13} + \frac{1}{13} + \frac{4}{13} = \frac{12}{13}$$

Or $P(x > -2) = 1 - P(x \leq -2) = 1 - \frac{1}{13} = \frac{12}{13}$

$\Rightarrow P(x\ at\ least\ 2) = P(x \geq 2) = P(x = 2) + P(x = 3) + P(x = 5)$

$$= \frac{2}{13} + \frac{1}{13} + \frac{4}{13} = \frac{7}{13}$$

Or $P(x \geq 2) = 1 - P(x < 2) = 1 - \frac{6}{13} = \frac{7}{13}$

$\Rightarrow P(x\ more\ than\ 2) = P(x > 2) = P(x = 3) + P(x = 5)$

$$= \frac{1}{13} + \frac{4}{13} = \frac{5}{13}$$

$$\Longrightarrow P(-1 < x \leq 3) = P(x = 0) + P(x = 2) + P(x = 3)$$

$$= \frac{3}{13} + \frac{2}{13} + \frac{1}{13} = \frac{6}{13}$$