Def. 5: **Continuous Random Variable**

A random variable X that may take on any value on a continuum is called a continuous random variable. In other words, a variable which takes all possible values between its limits say $\alpha \; and \; \beta$ is called a continuous random variable.

Ex.16: A light bulb is turned on continuously and we observed the time X until it burns out.

Ex.17: The length of school children is an example of a continuous random variable.

Def. 6: **Probability Density Function**

The probability density function of a continuous random variable X is a function f which the property that the area under the curve of this function corresponding to any interval is equal to the probability that X will have a value in this interval. Then $f(x)$ is called the probability density function (p.d.f.) of X.

Properties of the p.d.f.:

1. $f(x) \geq 0 \quad \forall x$
2. $\int_{\forall x} f(x)dx = 1$
3. For any interval (a, b)

   $$P(a \leq x \leq b) = \int_a^b f(x)dx = F(b) - F(a)$$

4. $P(x = a) = 0 \quad , a \; constant$

Ex.18: Let the random variable X be the length of life of an electron tube, with space $R = \{0 \leq x < \infty\}$. Suppose that a reasonable probability model for X is given by the p.d.f.

$$f(x) = \begin{cases} \frac{1}{100} e^{-x/100} & x > 0 \\ 0 & o.w. \end{cases}$$

Solution:

We have

1. $f(x) > 0$
2. $\int_{all\ x} f(x)dx = 1$

1. Prove (2)

$$\int\limits_{all\ x} f(x)dx = \int\limits_0^\infty \frac{1}{100} e^{-x/100}\, dx = -\int\limits_0^\infty \frac{-1}{100} e^{-x/100}\, dx$$
$$= -\left[e^{-x/100}\right]_0^\infty = -[e^{-\infty} - e^0] = 1$$

2. Find the probability that this electron tube lasts more than 100 hours.

$$P(x > 100) = \int\limits_{100}^\infty \frac{1}{100} e^{-x/100}\, dx = -\int\limits_{100}^\infty \frac{-1}{100} e^{-x/100}\, dx$$
$$= -\left[e^{-x/100}\right]_{100}^\infty = -\left[e^{-\infty} - e^{-100/100}\right] = -[0 - e^{-1}]$$
$$= 0.368$$

<u>Ex.19</u>: Let the random variable X have the p.d.f.:

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

Find

1. $P(1/2 < x < 3/4)$
2. $P(-1/2 < x < 1/2)$

Solution:

$$1. P(1/2 < x < 3/4) = \int\limits_{1/2}^{3/4} f(x)dx = \int\limits_{1/2}^{3/4} 2x\, dx = \left[\frac{2\, x^2}{2}\right]_{1/2}^{3/4}$$
$$= [\, x^2]_{1/2}^{3/4} = [(3/4)^2 - (1/2)^2] = \frac{5}{16}$$

$2. P(-1/2 < x < 1/2)$
$$= \int\limits_{-1/2}^{1/2} f(x)dx = \int\limits_{-1/2}^{0} 2x\, dx + \int\limits_{0}^{1/2} 2x dx = 0 + \left[\frac{2\, x^2}{2}\right]_0^{1/2}$$
$$= \frac{1}{4}$$

<u>Ex.20</u>: Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & o.w. \end{cases}$$

i. What is the value of the constant C?
ii. Find the probability that $x > 1$?
iii. Find the probability that $x > 0$?

Solution:

i. In order to find the value of C, we use the property that $f(x)$ is a p.d.f., that is

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{0}^{2} C(4x - 2x^2)\, dx = 1 \Rightarrow C\left[\frac{4\,x^2}{2} - \frac{2\,x^3}{3}\right]_0^2 = 1$$

$$\frac{8}{3}C = 1$$

$$\therefore\ C = \frac{3}{8}$$

$$f(x) = \begin{cases} \frac{3}{8}(4x - 2x^2) & 0 < x < 2 \\ 0 & o.w. \end{cases}$$

ii. $P(x > 1) = \int_1^2 \frac{3}{8}(4x - 2x^2)\, dx = \frac{3}{8}\left[\frac{4\,x^2}{2} - \frac{2\,x^3}{3}\right]_1^2 = \frac{1}{2}$

iii. $P(x > 0) = \int_0^2 \frac{3}{8}(4x - 2x^2)\, dx = \frac{3}{8}\left[\frac{4\,x^2}{2} - \frac{2\,x^3}{3}\right]_0^2 = 1$

<u>Def. 7</u>: **Cumulative Distribution Function**

The cumulative probability function of a continuous random variable is denoted by $F(x)$ and is defined by:

$F(x) = P(X \leq x)$ For any real number x.

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

We may also refer to the distribution function as we have seen it in the case of discrete random variable by the cumulative distribution function (c.d.f.) or the cumulative probability function.

Properties of the c.d.f.:

1. $\lim_{x\to\infty} F(x) = 1$
2. $\lim_{x\to-\infty} F(x) = 0$
3. $F(x_1) \le F(x_2) \quad if\ x_1 < x_2$
4. $P(a \le x \le b) = F(b) - F(a)$

Note:

$$f(x) = \frac{d\,F(x)}{d\,x} \qquad \forall x$$

<u>Ex.21</u>: Let the random variable X have the p.d.f.:

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

1. Show that $f(x)$ p.d.f.?
2. Find the c.d.f.?
3. Find$P(x > 0), P(x < 1), P(x > -2), P\left(-2 < x < \frac{1}{2}\right)$, $P\left(0 < x < \frac{1}{2}\right)$

Solution:

1. $\int_{all\ x} f(x)\,dx = 1 \implies \int_0^1 2x\,dx = 2\left[\frac{x^2}{2}\right]_0^1 = 2\left[\frac{1}{2} - 0\right] = 1$

2. $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$

$$= \int_0^x 2t\,dt = 2\left[\frac{t^2}{2}\right]_0^x = 2\left[\frac{x^2}{2} - 0\right] = x^2$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

3. $P(x > 0) = P(0 < x < 1) = \int_0^1 f(x)dx = \int_0^1 2x\, dx = 2\left[\frac{x^2}{2}\right]_0^1 = 1$

$$P(x < 1) = P(0 < x < 1) = \int_0^1 f(x)dx = \int_0^1 2x\, dx = 2\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$P(x > -2) = P(x > 0) = P(0 < x < 1) = 1$$

$$P\left(-2 < x < \frac{1}{2}\right) = P(-2 < x < 0) + P\left(0 < x < \frac{1}{2}\right)$$

$$= 0 + \int_0^{1/2} f(x)dx = \int_0^{1/2} 2x\, dx = 2\left[\frac{x^2}{2}\right]_0^{1/2}$$

$$= \frac{1}{4} - 0 = \frac{1}{4}$$

$$P\left(0 < x < \frac{1}{2}\right) = \int_0^{1/2} f(x)dx = \int_0^{1/2} 2x\, dx = 2\left[\frac{x^2}{2}\right]_0^{1/2} = \frac{1}{4}$$

Or

$$P\left(0 < x < \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F(0) = \left(\frac{1}{2}\right)^2 - (0)^2 = \frac{1}{4}$$

Ex.22: Let the random variable X have the probability density function:

$$f(x) = \begin{cases} C\,x^2 & -1 < x < 1 \\ 0 & o.w. \end{cases}$$

1. Determine the value of the constant C.
2. Find the cumulative distribution function.
3. Find $P(0 \le x \le 1), P(0 \le x \le 3), P\left(x = \frac{1}{2}\right), P\left(-1 \le x \le \frac{1}{2}\right), P(|x| \le 2)$

Solution:

1. $\int_{-\infty}^{\infty} f(x)\, dx = 1 \;\Rightarrow\; \int_{-1}^{1} C\, x^2\, dx = 1$

$$C\left[\frac{x^3}{3}\right]_{-1}^{1} = 1 \;\Rightarrow\; C\left[\frac{1}{3}+\frac{1}{3}\right] = 1$$

$$C\left[\frac{2}{3}\right] = 1$$

$$\therefore\; C = \frac{3}{2}$$

$$f(x) = \begin{cases} \frac{3}{2}\,x^2 & -1 < x < 1 \\ 0 & o.w. \end{cases}$$

2. $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$

$$= \int_{-1}^{x} \frac{3}{2}\, t^2\, dt = 1/2\,(x^3+1)$$

$$F(x) = \begin{cases} 0 & x < -1 \\ 1/2\,(x^3+1) & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

3. We can use this c.d.f. $F(x)$ to compute the following probabilities:

a) $P(0 \le x \le 1) = F(1) - F(0)$

$$= 1 - 1/2\,(0+1) = 1/2$$

b) $P(0 \le x \le 3) = F(3) - F(0)$

$$= 1 - 1/2\,(0+1) = 1/2$$

c) $P\left(x = \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{2}\right) = 0$

d) $P\left(-1 \le x \le \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F(-1)$

$$= 1/2\,((1/2)^3+1) - 0 = 9/16$$

e) $P(|x| \le 2) = P(-2 \le x \le 2) = F(2) - F(-2) = 1 - 0 = 1$

Ex.23: Let X be a r.v. with distribution function given by:

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

1. Find the p.d.f. $f(x)$.
2. Find $P(x \leq 0.7), P(x = 1) and\, P(x > 0.5)$.

Solution:

1. $f(x) = \frac{d\, F(x)}{d\, x} = \frac{d\, (x^3)}{dx} = 3x^2$

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

2. $P(x \leq 0.7) = F(0.7) = (0.7)^3 = 0.343$

$P(x = 1) = 0$ since X continuous random variable.

$P(x > 0.5) = 1 - P(x \leq 0.5) = 1 - F(0.5) = 1 - (0.5)^3 = 0.875$

Ex.24: Suppose the continuous distribution function is:

$$F(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-y} & y \geq 0 \end{cases}$$

1. Find the p.d.f. $f(y)$.
2. Find $P(1 < Y < 3), P(Y > 4) and\, P(Y < 2)$.

Solution:

1. $f(x) = \frac{d\, F(y)}{d\, y} = \frac{d\, (1 - e^{-y})}{dy} = e^{-y}$

$$f(x) = \begin{cases} e^{-y} & y > 0 \\ 0 & o.w. \end{cases}$$

2. $P(1 < Y < 3) = \int_1^3 e^{-y}\, dy = -\int_1^3 - e^{-y}\, dy = -[\, e^{-y}]_1^3$

$$= -[e^{-3} - e^{-1}] = e^{-1} - e^{-3} = 0.318$$

$$P(Y > 4) = \int_4^\infty e^{-y}\, dy = -\int_4^\infty - e^{-y}\, dy = -[\, e^{-y}]_4^\infty$$

$$= -[e^{-\infty} - e^{-4}] = e^{-4} - 0 = e^{-4} = 0.018$$

$$P(Y < 2) = \int_0^2 e^{-y}\, dy = -\int_0^2 - e^{-y}\, dy = -[\, e^{-y}]_0^2$$

$$= -[e^{-2} - e^{0}] = 1 - e^{-2} = 1 - 0.135 = 0.865$$

Problems

1. Let $P(x)$ be the p.m.f. of a random variable x. Find the distribution function $F(x)$ and sketch its graph, where
   a) $P(x) = 1 \qquad x = 3$
   b) $P(x) = \frac{1}{2} \qquad x = 1,2$
   c) $P(x) = \frac{x}{15} \qquad x = 1,2,3,4,5$
   d) $P(x) = (\frac{1}{4})(\frac{3}{4})^{x} \quad x = 0,1,2, \ldots.$

2. The random variable x has the probability mass function

$$P(x) = \begin{cases} 1/3 & x = 0,1,2 \\ 0 & o.w. \end{cases}$$

   a) Show that $P(x)$ is probability mass function?
   b) What is the distribution function of x?

3. For the following function

$$P(x) = \begin{cases} a\,(\frac{1}{3})^x & x = 1,2,3,\ldots \\ 0 & o.w. \end{cases}$$

   a) Find the value of a so that $P(x)$ satisfies the conditions of being a probability mass function for a r.v. x.
   b) Find the c.d.f. of x?

4. If

$$F(x) = \begin{cases} 0 & x < 0 \\ 10/28 & 0 \le x < 1 \\ 25/28 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

   a) Find the p.m.f.?
   b) Find $P(-1 < x < 2)$?
   c) Find $P(x - 3 \ge -1)$?

5. If X is a r.v. having the following p.m.f.:

X	-2	1	3	4
$P(x)$	a^2	$2a^2$	a	a

a) Show that $a = \frac{1}{3}$.
b) Find the c.d.f.
c) Find

$P(x > 4), P(x < 2), P(x = 2), P(x = 1), P(x = -2), P(0 \le x \le 2)$.

6. Suppose that x is a continuous random variable whose p.d.f. is given by:

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & o.w. \end{cases}$$

   a) What is the value of the constant C?
   b) Find the probability that $x > 1$?

c) Find the probability that $x > 0$?

7. Let x is a continuous random variable whose c.d.f. is given by:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-3x} & 0 \leq x < \infty \\ 1 & x \longrightarrow \infty \end{cases}$$

a) Find the p.d.f.?

b) Find $P(0 \leq x \leq 1), P(0 \leq x \leq \frac{1}{2})$?

8. Let $f(x)$ be the p.d.f. of a random variable x. Find the distribution function $F(x)$, where

a) $f(x) = 1/x^2$ $\quad 1 < x < \infty$

b) $f(x) = 2x$ $\quad 0 < x < 1$

c) $f(y) = 2(1 - y)$ $\quad 0 < y < 1$

9. Let the r.v. x have the following p.d.f.:

$$f(x) = \begin{cases} Cx^4 & -1 < x < 1 \\ 0 & o.w. \end{cases}$$

a) What is the value of the constant C?

b) Find the distribution function $F(x)$

c) Find $P(|x| \leq 0.2)$?

d) Find $P(-2 < x < 0.2)$?