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قسم الاحصاء

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Probability Distributions

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Some Discrete Probability Distributions

1. Discrete Uniform Distribution

Suppose that a random variable X takes on a finite set of values $\{1,2,3,\ldots,n\}$. The r.v. X is said to have a discrete uniform distribution with parameter n, and its probability mass function is given by:

$$p(x,n)=\begin{cases}1/n & x=1,2,3,\ldots,n\\ 0 & otherwise\end{cases}$$

Where (n) is positive integer.

1. $0\leq p(x,n)\leq 1$
2. $\sum_{\forall\, x} p(x)=1$

   $\sum_{x=1}^{n}\frac{1}{n}=\frac{1}{n}\sum_{x=1}^{n}1=\frac{n}{n}=1$

3. The **mean** of the r.v. $X\sim Du(n)$ can be obtain as follows

$$\mu_x=E(x)=\frac{n+1}{2}$$

4. The **variance** of the r.v. $X\sim Du(n)$ can be obtain as follows

$$var(x)=\sigma_x^2=E(x^2)-(E(x))^2=\frac{n^2-1}{12}$$

5. The Cumulative Distribution Function of X is given by:

$$F(x)=P(X\leq x)=\sum_{X_i\leq x}p(x_i)$$

<u>**Ex.1**</u>**:** suppose that we numbered five balls from 1 to 5 and one ball is selected at random. Let X be the number on the selected ball. Then

1. Write the p.m.f. and the c.d.f. of X.
2. Find the mean and the variance of X.

Solution:

1.$X\sim Du(n)\rightarrow X\sim Du(5)$

$$p(x)=\begin{cases}1/5 & x=1,2,3,4,5\\ 0 & otherwise\end{cases}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/5 & 1 \le x < 2 \\ 2/5 & 2 \le x < 3 \\ 3/5 & 3 \le x < 4 \\ 4/5 & 4 \le x < 5 \\ 5/5 = 1 & x \ge 5 \end{cases}$$

2. $\mu_x = \frac{n+1}{2} = \frac{5+1}{2} = \frac{6}{2} = 3$

$$var(x) = \sigma_x^2 = \frac{n^2-1}{12} = \frac{5^2-1}{12} = \frac{24}{12} = 2$$

<u>Ex.2</u>: suppose a die is tossed once. Let the r.v. X denote the number that appears. Then

1. Write the p.m.f. and the c.d.f. of X.
2. Find the mean (μ_x) and the variance (σ_x^2).
3. Find $P(x \ge 4), P(2 < x \le 4), P(x < 1)$.
4. Find $P(2 \le x \le 5), P(1 \le x \le 4)$ by using c.d.f.

Solution:

1.$X \sim Du(n) \rightarrow X \sim Du(n = 6)$

$$p(x) = \begin{cases} 1/6 & x = 1,2,3,4,5,6 \\ 0 & otherwise \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \le x < 2 \\ 2/6 & 2 \le x < 3 \\ 3/6 & 3 \le x < 4 \\ 4/6 & 4 \le x < 5 \\ 5/6 & 5 \le x < 6 \\ 6/6 & x \ge 6 \end{cases}$$

2.$\mu_x = \frac{n+1}{2} = \frac{6+1}{2} = \frac{7}{2}$

$$var(x) = \sigma_x^2 = \frac{n^2 - 1}{12} = \frac{6^2 - 1}{12} = \frac{35}{12}$$

3.

$\Longrightarrow P(x \ge 4) = P(x = 4) + P(x = 5) + P(x = 6)$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$$

$$\Longrightarrow P(2 < x \leq 4) = P(x = 3) + P(x = 4)$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$\Longrightarrow P(x < 1) = 0$$

4.

$$\Longrightarrow P(2 \leq x \leq 5) = F(5) - F(2) = \frac{5}{6} - \frac{2}{6} = \frac{3}{6}$$

$$\Longrightarrow P(1 \leq x \leq 4) = F(4) - F(1) = \frac{3}{6} - \frac{1}{6} = \frac{2}{6}$$

2. Bernoulli Distribution

Suppose that a trial whose outcome can be classified as either a success or a failure is performed. Let X be a r.v. taking the value 1 if the outcome is a success and 0 if it is a failure. Then the r.v. X is said to have a Bernoulli distribution with parameter p, and its p.m.f. is given by:

$$p(x,p) = \begin{cases} p^x\,(1-p)^{1-x} & x = 0,1 \\ 0 & otherwise \end{cases}$$

Where $0 \le p \le 1$ and $(1-p) = q, 0 \le q \le 1$ then

1. $0 \le p(x,p) \le 1$
2. $\sum_{\forall\, x} p(x) = 1$

$$\sum_{x=0}^{1} p^x\,(1-p)^{1-x} = p^0(1-p)^{1-0} + p^1(1-p)^{1-1}$$

$$= 1 - p + p = 1$$

3. The **mean** of the r.v. $X \sim B(1,p)$ can be obtain as follows

$$\mu_x = E(x) = p$$

4. The **variance** of the r.v. $X \sim B(1,p)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = pq$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Ex.3: consider the experiment of tossing a fair die, and suppose that obtaining a 2 or 3 is considered a success.

1. Write the p.m.f. and the c.d.f. of X.
2. Find the mean (μ_x) and the variance (σ_x^2).

Solution:

1.$X \sim B(1,p)$

$S = \{1,2,3,4,5,6\}$

Success $\Longrightarrow \{2,3\} \Longrightarrow x = 1 \;\Longrightarrow p(x=1) = \frac{2}{6} = \frac{1}{3} = p$

Failure $\Longrightarrow \{1,4,5,6\} \Longrightarrow x = 0 \;\Longrightarrow p(x=0) = \frac{4}{6} = \frac{2}{3} = 1-p$

$$p(x) = \begin{cases} \left(\frac{1}{3}\right)^{x} \left(\frac{2}{3}\right)^{1-x} & x = 0,1 \\ 0 & otherwise \end{cases}$$

$$p(x) = \begin{cases} 1/3 & x = 1 \\ 2/3 & x = 0 \\ 0 & o.w. \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 2/3 & 0 \le x < 1 \\ 2/3 + 1/3 = 1 & x \ge 1 \end{cases}$$

2.

$$\mu_x = E(x) = p = \frac{1}{3}$$

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = pq = \frac{1}{3} * \frac{2}{3} = \frac{2}{9}$$

Ex.4: suppose a coin is tossed. Let the random variable X take the value 1 if the outcome is a head (success) and 0 if it is a tail (failure).

1. Write the p.m.f. and the c.d.f. of X.
2. Find the mean (μ_x) and the variance (σ_x^2).
3. Find $P(x=0), P(x=1)$.

Solution:

1.$X \sim B(1,p)$

$S = \{H, T\}$

Success $\Longrightarrow \{H\} \Longrightarrow x = 1 \;\Longrightarrow p(x=1) = \frac{1}{2} = p$

Failure $\Longrightarrow \{T\} \Longrightarrow x = 0 \;\Longrightarrow p(x=0) = \frac{1}{2} = 1-p$

$$p(x) = \begin{cases} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{1-x} & x = 0,1 \\ 0 & otherwise \end{cases}$$

$$p(x) = \begin{cases} 1/2 & x = 1 \\ 1/2 & x = 0 \\ 0 & o.w. \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 \le x < 1 \\ 1/2 + 1/2 = 1 & x \ge 1 \end{cases}$$

2.

$$\mu_x = E(x) = p = \frac{1}{2}$$

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = pq = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$

3.

$$\Rightarrow P(x = 0) = \frac{1}{2}$$

$$\Rightarrow P(x = 1) = \frac{1}{2}$$

Ex.5: suppose a ball is drawn at random from an urn containing 20 red and 30 white balls. Let the random variable X take the value 1 if the drawn ball is red (success) and 0 if it is white (failure).

1. Find the p.m.f. and the c.d.f. of X.
2. Find the mean (μ_x) and the variance (σ_x^2).

Solution:

1.$X \sim B(1,p)$

Success $\Rightarrow \{red\} \Rightarrow x = 1 \Rightarrow p(x = 1) = \frac{20}{50} = \frac{2}{5} = p$

Failure $\Rightarrow \{white\} \Rightarrow x = 0 \Rightarrow p(x = 0) = \frac{30}{50} = \frac{3}{5} = 1 - p$

$$p(x) = \begin{cases} \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{1-x} & x = 0,1 \\ 0 & otherwise \end{cases}$$

$$p(x)=\begin{cases}2/5 & x=1\\ 3/5 & x=0\\ 0 & o.w.\end{cases}$$

$$F(x)=\begin{cases}0 & x<0\\ 3/5 & 0\le x<1\\ 3/5+2/5=1 & x\ge 1\end{cases}$$

2. $\mu_x = E(x) = p = \frac{2}{5}$, $var(x) = \sigma_x^2 = pq = \frac{2}{5} * \frac{3}{5} = \frac{6}{25}$

<u>Ex.6</u>: In a survey study, 60 people agreed with a question and 40 people disagreed. Suppose one person is selected at random from the survey. Let the random variable X take the value 1 if the selected person agrees with the question and 0 if the person disagrees. Find the probability mass function and the cumulative distribution function of X. Also, find the mean and the variance of X.

Solution:

$$X \sim B(p) \quad , \quad p=\frac{60}{100}=0.6 \quad ,q=1-p=1-0.6=0.4$$

$$\Longrightarrow p(x)=\begin{cases}(0.6)^x\,(0.4)^{1-x} & x=0,1\\ 0 & otherwise\end{cases}$$

$$p(x)=\begin{cases}0.6 & x=1\\ 0.4 & x=0\\ 0 & o.w.\end{cases}$$

$$\Longrightarrow F(x)=\begin{cases}0 & x<0\\ 0.4 & 0\le x<1\\ 0.4+0.6=1 & x\ge 1\end{cases}$$

$$\Longrightarrow \mu_x = E(x) = p = 0.6$$

$$\Longrightarrow var(x) = \sigma_x^2 = pq = (0.6)*(0.4) = 0.24$$

3. Binomial Distribution

Consider a series of $\boldsymbol{n}$ independent Bernoulli trials, where $\boldsymbol{n}$ is a finite integer, and the probability of success p in each trial is constant. Then $q = 1 - p$ is the probability of the failure in each trial.

Let X denote the number of successes in $\boldsymbol{n}$ trials. Then X can take the values $(0,1,2,\dots,n)$ with non-zero probabilities, and X is said to be a binomial random variable with parameters $\boldsymbol{n}$ $\boldsymbol{and}$ $\boldsymbol{p}$. Its p.m.f. is given by:

$$p(x,n,p) = \begin{cases} C_x^n \ p^x \ (1-p)^{n-x} & x = 0,1,2,\dots,n \\ 0 & otherwise \end{cases}$$

1. $0 \le p(x,n,p) \le 1$
2. $\sum_{\forall x} p(x,n,p) = 1$

$$\sum_{\forall x} p(x,n,p) = \sum_{x=0}^{n} C_x^n \ p^x \ (1-p)^{n-x}$$

$$= C_0^n \ p^0 \ (1-p)^{n-0} + C_1^n \ p^1 \ (1-p)^{n-1} + \cdots + C_n^n \ p^n \ (1-p)^{n-n}$$

$$= q^n + npq^{n-1} + \cdots + p^n = (q+p)^n$$

But $q + p = 1$

$$\therefore \sum_{x=0}^{n} C_x^n \ p^x \ (1-p)^{n-x} = 1$$

3. The **mean** of the r.v. $X \sim Bin(n,p)$ can be obtain as follows

$$\mu_x = E(x) = np$$

4. The **variance** of the r.v. $X \sim Bin(n,p)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = npq$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} C_k^n \ p^k \ (1-p)^{n-k} & 0 \le x < n \\ 1 & x \ge n \end{cases}$$

Ex.7: a family has 4 children. Assume that the birth of each sex is equally likely. Let X denote the number of boys in the family.

1. Find the p.m.f. and the c.d.f. of X.
2. Find the mean (μ_x) and the variance (σ_x^2).

Solution:

1. $X \sim Bin(n,p) \quad , \quad n = 4 \quad , p = \frac{1}{2} \quad , q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$P(B) = P(G) = \frac{1}{2}$

$$p(x) = \begin{cases} C_x^4 \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} & x = 0,1,2,3,4 \\ 0 & otherwise \end{cases}$$

$$P(x = 0) = C_0^4 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{4-0} = \frac{1}{16}$$

$$P(x = 1) = C_1^4 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{4-1} = \frac{4}{16}$$

$$P(x = 2) = C_2^4 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} = \frac{6}{16}$$

$$P(x = 3) = C_3^4 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{4-3} = \frac{4}{16}$$

$$P(x = 4) = C_4^4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{4-4} = \frac{1}{16}$$

$$p(x) = \begin{cases} 1/16 & x = 0 \\ 4/16 & x = 1 \\ 6/16 & x = 2 \\ 4/16 & x = 3 \\ 1/16 & x = 4 \\ 0 & o.w. \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1/16 & 0 \le x < 1 \\ 5/16 & 1 \le x < 2 \\ 11/16 & 2 \le x < 3 \\ 15/16 & 3 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

$2. \mu_x = E(x) = np = 4 * \frac{1}{2} = 2 \quad , \quad var(x) = \sigma_x^2 = npq = 4 * \frac{1}{2} * \frac{1}{2} = 1$

<u>Ex.8</u>: a machine produces a certain item with a defective rate of 0.05. A random sample of 6 items is selected from the output of this machine. Let X be the number of defectives in the sample. Find the probability mass function of X.

Solution:

$X \sim Bin(6,0.05) \;, p = 0.05 \;, q = 1 - p = 1 - 0.05 = 0.95$

$$p(x) = \begin{cases} C_x^6 \; (0.05)^x \, (0.95)^{6-x} & x = 0,1,2,3,4,5,6 \\ 0 & otherwise \end{cases}$$

$P(x = 0) = C_0^6 \; (0.05)^0 \, (0.95)^{6-0} = 0.7351$

$P(x = 1) = C_1^6 \; (0.05)^1 \, (0.95)^{6-1} = 0.2321$

$P(x = 2) = C_2^6 \; (0.05)^2 \, (0.95)^{6-2} = 0.0305$

$P(x = 3) = C_3^6 \; (0.05)^3 \, (0.95)^{6-3} = 0.00214$

$P(x = 4) = C_4^6 \; (0.05)^4 \, (0.95)^{6-4} = 0.000084$

$P(x = 5) = C_5^6 \; (0.05)^5 \, (0.95)^{6-5} = 0.0000018$

$P(x = 6) = C_6^6 \; (0.05)^6 \, (0.95)^{6-6} = 0.000000016$

$$p(x) = \begin{cases} 0.7351 & x = 0 \\ 0.2321 & x = 1 \\ 0.0305 & x = 2 \\ 0.00214 & x = 3 \\ 0.000084 & x = 4 \\ 0.0000018 & x = 5 \\ 0.000000016 & x = 6 \\ 0 & o.w. \end{cases}$$

<u>Ex.9</u>: suppose that a coin is tossed 7 times. Let the r.v. X denote the tails obtained. Find the probability of getting:

1. exactly 2 tails 2. At least 4 tails 3. At least one tail

4. $P(4 \le x \le 6)$ 5. $P(x > 5)$

Solution:

$X \sim Bin(7,0.5) \quad ,p = 0.5, \; q = 1 - p = 1 - 0.5 = 0.5$

$$p(x) = \begin{cases} C_x^7 \; (0.5)^x \, (0.5)^{7-x} & x = 0,1,2,3,4,5,6,7 \\ 0 & otherwise \end{cases}$$

1. $P(x = 2) = C_2^7 \; (0.5)^2 \, (0.5)^{7-2} = \frac{7!}{2!*5!} \; (0.5)^2 \, (0.5)^5 = 0.1641$

2. $P(x \ge 4) = P(x = 4) + P(x = 5) + P(x = 6) + P(x = 7)$

$$= C_4^7 \; (0.5)^4 \, (0.5)^{7-4} + C_5^7 \; (0.5)^5 \, (0.5)^{7-5} + C_6^7 \; (0.5)^6 \, (0.5)^{7-6} + C_7^7 \; (0.5)^7 \, (0.5)^{7-7}$$

$=\frac{7!}{4!*3!} \; (0.5)^4 \, (0.5)^3 + \frac{7!}{5!*2!} \; (0.5)^5 \, (0.5)^2 + \frac{7!}{6!*1!} \; (0.5)^6 \, (0.5)^1 + \frac{7!}{7!*0!} \; (0.5)^7 \, (0.5)^0 = 0.5$

3. $P(x \ge 1) = 1 - P(x = 0) = 1 - C_0^7 \; (0.5)^0 \, (0.5)^{7-0}$

$$= 1 - \frac{7!}{0! * 7!} \; (0.5)^0 \, (0.5)^7 = 0.9922$$

4. $P(4 \le x \le 6) = P(x = 4) + P(x = 5) + P(x = 6)$

$$= C_4^7 \; (0.5)^4 \, (0.5)^{7-4} + C_5^7 \; (0.5)^5 \, (0.5)^{7-5} + C_6^7 \; (0.5)^6 \, (0.5)^{7-6}$$

$= \frac{7!}{4!*3!} \; (0.5)^4 \, (0.5)^3 + \frac{7!}{5!*2!} \; (0.5)^5 \, (0.5)^2 + \frac{7!}{6!*1!} \; (0.5)^6 \, (0.5)^1$

$$= 0.4922$$

5. $P(x > 5) = P(x = 6) + P(x = 7)$

$$= C_6^7 \; (0.5)^6 \, (0.5)^{7-6} + C_7^7 \; (0.5)^7 \, (0.5)^{7-7} = 0.0625$$

4. Poisson Distribution

The poisson distribution appears in many natural and physical phenomena, such as

1. The number of misprints per page in a large text.
2. The number of accidents per unit of time (hour, day, week, or month) on a highway.
3. The number of telephone calls per unit of time received at some switchboard.

A random variable X is said to have a poisson distribution with parameter λ if its p.m.f. is given by:

$$p(x,\lambda)=\begin{cases}\dfrac{e^{-\lambda}\,\lambda^{x}}{x!} & x=0,1,\dots\dots\\ 0 & otherwise\end{cases}$$

1. $0\le p(x,\lambda)\le 1$
2. $\sum_{\forall x}p(x,\lambda)=1$

$$\sum_{x=0}^{\infty}\frac{e^{-\lambda}\,\lambda^{x}}{x!}=e^{-\lambda}\,\sum_{x=0}^{\infty}\frac{\lambda^{x}}{x!}$$

Recall that $e^{\lambda}\;=\;\sum_{x=0}^{\infty}\frac{\lambda^{x}}{x!}$

$$\therefore\;\sum_{x=0}^{\infty}\frac{e^{-\lambda}\,\lambda^{x}}{x!}=e^{-\lambda}\;e^{\lambda}\;=e^{0}\;=1$$

3. The **mean** of the r.v. $X\sim Po(\lambda),\lambda>0$ can be obtain as follows

$$\mu_x=E(x)=\lambda$$

4. The **variance** of the r.v. $X\sim Po(\lambda),\lambda>0$ can be obtain as follows

$$var(x)=\sigma_x^2\;=E(x^2)-(E(x))^2=\lambda$$

5. The Cumulative Distribution Function of X is given by:

$$F(x)=P(X\le x)\;=\begin{cases}0 & x<0\\ \displaystyle\sum_{k=0}^{x}\frac{e^{-\lambda}\,\lambda^{k}}{k!} & 0\le x<\infty\\ 1 & x\to\infty\end{cases}$$

Or

$$F(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} \frac{e^{-\lambda}\, \lambda^{k}}{k!} & x \ge 0 \end{cases}$$

<u>Ex.10</u>: the average number of claims filed against an insurance company is 2 claims per day. What is the probability that, on any given day

1. Exactly one claim is filed against the insurance company?
2. No claim is filed against the insurance company?
3. Exactly three claims are filed against the insurance company?

Solution:

$X \sim Po(2) \quad ,\lambda = 2$

$$p(x,\lambda) = \begin{cases} \frac{e^{-2}\; 2^{x}}{x!} & x = 0,1,\dots\dots \\ 0 & otherwise \end{cases}$$

1. $P(x = 1) = \frac{e^{-2}\; 2^{1}}{1!} = 0.2707$

2. $P(x = 0) = \frac{e^{-2}\; 2^{0}}{0!} = 0.1353$

3. $P(x = 3) = \frac{e^{-2}\; 2^{3}}{3!} = 0.1804$

<u>Ex.11</u>: an office switchboard receives phone calls at an average rate of 3 calls per minute. Let X be the number of phone calls received in one minute.

1. Find the p.m.f. and the c.d.f. of X.
2. Find the mean (μ_x) and the variance (σ_x^2).

Solution:

1. $X \sim Po(3) \quad ,\lambda = 3$

$$p(x,\lambda) = \begin{cases} \frac{e^{-3}\; 3^{x}}{x!} & x = 0,1,\dots\dots \\ 0 & otherwise \end{cases}$$

$$F(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} \frac{e^{-\lambda}\,\lambda^k}{k!} & x \ge 0 \end{cases}$$

$$F(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} \frac{e^{-3}\,3^k}{k!} & x \ge 0 \end{cases}$$

2. $\mu_x = E(x) = \lambda = 3$, $var(x) = \sigma_x^2 = \lambda = 3$

Ex.12: A factory produces 200 lamps per day. The number of defective lamps per day has an average of 20. Let the r.v. X denote the number of defective lamps produced per day. Answer the following:

1. Write the probability mass function of X.
2. Find the mean (μ_x) and the variance (σ_x^2).
3. Find the probability that there is no defective lamp.
4. Find the probability that there are two defective lamps.
5. Find the probability of having at most two defective lamps.
6. Find the probability of having at least two defective lamps.

Solution:

1. $X \sim Po(\lambda)$ $, \lambda = 20$

$$p(x,\lambda) = \begin{cases} \frac{e^{-20}\,20^x}{x!} & x = 0,1,\dots\dots \\ 0 & otherwise \end{cases}$$

2. $\mu_x = E(x) = \lambda = 20$, $var(x) = \sigma_x^2 = \lambda = 20$

3. $P(x = 0) = \frac{e^{-20}\,20^0}{0!} = e^{-20} = 0.000000002$

4. $P(x = 2) = \frac{e^{-20}\,20^2}{2!} = 0.00000041$

5. $P(x \le 2) = P(x = 2) + P(x = 1) + P(x = 0)$

$$= \frac{e^{-20}\,20^2}{2!} + \frac{e^{-20}\,20^1}{1!} + \frac{e^{-20}\,20^0}{0!} = 0.00000045$$

6. $P(x \ge 2) = 1 - P(x < 2) = 1 - [P(x = 0) + P(x = 1)]$

$$= 1 - \left[\frac{e^{-20}\ 20^{0}}{0!} + \frac{e^{-20}\ 20^{1}}{1!}\right] = 0.999$$

Remark: the poisson distribution can be derived as a limiting case of the binomial distribution under the following conditions:

1. The number of trials n is very large $(n \to \infty)$.
2. The probability of success p of each trial is very small $(p \to 0)$.

<u>Ex.13:</u> A factory produces 1000 lamps per day. The probability that a lamp is defective is very small $p = 0.01$. Let X be the number of defective lamps. Write the p.m.f of X.

Solution:

$\Longrightarrow\ X \sim Bin(1000{,}0.01) \quad ,p = 0.01,\ q = 1 - p = 1 - 0.01 = 0.99$

$$p(x) = \begin{cases} C_{x}^{1000}\ (0.5)^{x}\ (0.5)^{1000-x} & x = 0,1,2,\dots\dots,1000 \\ 0 & otherwise \end{cases}$$

$\Longrightarrow\ X \sim Po(\lambda) \quad ,\lambda = 0.01 * 1000 = 10$

$$p(x,\lambda) = \begin{cases} \dfrac{e^{-10}\ 10^{x}}{x!} & x = 0,1,\dots\dots \\ 0 & otherwise \end{cases}$$

5. Geometric Distribution

Consider an experiment of performing independent trials until we get the first success. The probability of each individual trial result in a success is p, Where $0 \leq p \leq 1$. Let X denotes the number of failures encountered before get the first success. Then the probability mass function (p.m.f.) of X is:

$$p(x,p) = \begin{cases} p\, q^x & x = 0,1,2,\dots\dots \\ 0 & otherwise \end{cases}$$

p (The parameter of the distribution) is the probability of success, and $q = 1 - p$ is the probability of failure.

1. $0 \leq p(x,p) \leq 1$
2. $\sum_{\forall x} p(x,p) = 1$

$$\sum_{x=0}^{\infty} p(x,p) = \sum_{x=0}^{\infty} p\, q^x = p \sum_{x=0}^{\infty} q^x$$

$$= p(q^0 + q^1 + q^2 + \cdots\cdots)$$

$$= p(1 + q + q^2 + \cdots\cdots)$$

$$= p\,\frac{1}{1-q} = \frac{p}{1-1+p} = \frac{p}{p} = 1$$

$$\therefore \sum_{x=0}^{\infty} p\, q^x = 1$$

3. The **mean** of the r.v. $X \sim G(p)$ can be obtain as follows

$$\mu_x = E(x) = \frac{q}{p}$$

4. The **variance** of the r.v. $X \sim G(p)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{q}{p^2}$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \leq x) = \begin{cases} 0 & x < 0 \\ 1 - q^{x+1} & 0 \leq x < \infty \\ 1 & x \to \infty \end{cases}$$

Or

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - q^{x+1} & x \geq 0 \end{cases}$$

Remark: we may define the Geometric distribution in a different way as the follows:

Let X denote the number of trials required to get the first success. Then the random variable X is said be a Geometric random variable if it has the following p.m.f:

$$p(x,p) = \begin{cases} p\, q^{x-1} & x = 1,2,\ldots\ldots \\ 0 & otherwise \end{cases}$$

1. $0 \leq p(x,p) \leq 1$
2. $\sum_{\forall x} p(x,p) = 1$

$$\sum_{x=1}^{\infty} p(x,p) = \sum_{x=1}^{\infty} p\, q^{x-1} = p \sum_{x=1}^{\infty} q^{x-1}$$

$$= p(q^0 + q^1 + q^2 + \cdots\cdots)$$

$$= p(1 + q + q^2 + \cdots\cdots)$$

$$= p\, \frac{1}{1-q} = \frac{p}{1-1+p} = \frac{p}{p} = 1$$

$\therefore \sum_{x=1}^{\infty} p\, q^{x-1} = 1$

3. The **mean** of the r.v. $X \sim G(p)$ can be obtain as follows

$$\mu_x = E(x) = \frac{1}{p}$$

4. The **variance** of the r.v. $X \sim G(p)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{q}{p^2}$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \leq x) = \begin{cases} 0 & x < 1 \\ 1 - q^x & 1 \leq x < \infty \\ 1 & x \to \infty \end{cases}$$

Or

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 - q^x & x \geq 1 \end{cases}$$

Ex.14: A coin is tossed until a head appears. Let the r.v. X denote the number of failure before the first head.

1. Give the distribution of X and write the p.m.f. of X.
2. Find the cumulative distribution function of X.
3. Find the mean (μ_x) and the variance (σ_x^2).
4. Find the probability that three tosses are needed.
5. Find the probability that at most three tosses are needed.
6. Find $P(2 \le x \le 8)$.

Solution:

1. $X \sim G(p) \quad , p = \frac{1}{2}, \; q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$$p(x,p) = \begin{cases} \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^x & x = 0,1,2, \dots \dots \\ 0 & otherwise \end{cases}$$

2.$F(x) = \begin{cases} 0 & x < 0 \\ 1 - q^{x+1} & x \ge 0 \end{cases}$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - \left(\frac{1}{2}\right)^{x+1} & x \ge 0 \end{cases}$$

3. $\mu_x = E(x) = \frac{q}{p} = \frac{1/2}{1/2} = 1 \qquad , \qquad var(x) = \sigma_x^2 = \frac{q}{p^2} = \frac{1/2}{(1/2)^2} = 2$

4. $P(x = 2) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{1}{8}$

5. $P(x \le 2) = P(x = 2) + P(x = 1) + P(x = 0)$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^0 = \frac{7}{8}$$

6. $P(2 \le x \le 8) = P(x = 2) + P(x = 3) + P(x = 4) + P(x = 5) + P(x = 6) + P(x = 7) + P(x = 8)$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + \cdots \cdots + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^8 = 0.247$$

Ex.15: A die is rolled until the number 6 appears. Let X denote the number of trials needed to obtain the first 6. Find the probability that

1. Write the p.m.f. and c.d.f. of X.
2. At least 4 trials are needed.
3. At most 4 trials are needed.
4. Exactly 4 trials are needed.

Solution:

1. $X \sim G(p) \quad ,p = \frac{1}{6}, \;\; q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

$$p(x,p) = \begin{cases} \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{x-1} & x = 1,2,\dots\dots \\ 0 & otherwise \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 - q^x & x \geq 1 \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 - \left(\frac{5}{6}\right)^x & x \geq 1 \end{cases}$$

2. $P(x \geq 4) = 1 - P(x < 4) = 1 - [P(x = 1) + P(x = 2) + P(x = 3)]$

$$= 1 - \left[\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{1-1} + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{2-1} + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{3-1}\right] = 0.579$$

3. $P(x \leq 4) = P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4)$

$$= \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{1-1} + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{2-1} + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{3-1} + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{4-1} = 0.518$$

4. $P(x = 4) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{4-1} = 0.097$

6. Negative Binomial Distribution

In the geometric distribution, the random variable X represents the number of failures before the first success. If we are interested in the r-th success instead of the first, then we use the negative binomial distribution, where the random variable X represents the number of failures before the r-th success. The probability mass function (p.m.f.) of X is given by:

$$p(x,r,p) = \begin{cases} C_x^{x+r-1}\ p^r\ q^x & x = 0,1,2,\dots\dots \\ 0 & otherwise \end{cases}$$

Thus $X \sim NB(r,p)$, where r and p is the distribution parameters, $r = 1,2,3,\cdots\quad and\ \ 0 \le p \le 1\ \ ,q = 1-p, p$ the probability of success.

1. $0 \le p(x,r,p) \le 1$
2. $\sum_{\forall x} p(x,r,p) = 1$

$$\sum_{x=0}^{\infty} p(x,r,p) = \sum_{x=0}^{\infty} C_x^{x+r-1}\ p^r\ q^x = p^r \sum_{x=0}^{\infty} C_x^{x+r-1}\ q^x$$

$$= p^r\left(1 + rq + \frac{r(r+1)}{2!}q^2 + \frac{r(r+1)(r+2)}{3!}q^3 + \cdots\cdots\right)$$

Recall that $\sum_{j=0}^{\infty} C_j^{r+j-1}\ x^j = (1-x)^{-r}$

$$= p^r(1-q)^{-r} = p^r(1-1+p)^{-r} = p^r p^{-r} = 1$$

$\therefore \sum_{x=0}^{\infty} C_x^{x+r-1}\ p^r\ q^x = 1$

3. The **mean** of the r.v. $X \sim NB(r,p)$ can be obtain as follows

$$\mu_x = E(x) = \frac{rq}{p}$$

4. The **variance** of the r.v. $X \sim NB(r,p)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{rq}{p^2}$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} C_k^{k+r-1} \ p^r \ q^k & 0 \le x < \infty \\ 1 & x \to \infty \end{cases}$$

Or

$$F(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} C_k^{k+r-1} \ p^r \ q^k & x \ge 0 \end{cases}$$

Remark:

In the geometric distribution, the random variable X represents the number of trials required to obtain the first success. If we are interested in the r-th success instead of the first, then we use the negative binomial distribution, where the random variable X represents the number of trials required to obtain the r-th success. Then the probability mass function (p.m.f.) of X is:

$$p(x, r, p) = \begin{cases} C_{r-1}^{x-1} \ p^r \ q^{x-r} & x = r, r+1, r+2, \dots \dots \\ 0 & otherwise \end{cases}$$

1. $0 \le p(x, r, p) \le 1$
2. $\sum_{\forall x} p(x, r, p) = 1$

$$\sum_{x=r}^{\infty} p(x, r, p) = \sum_{x=r}^{\infty} C_{r-1}^{x-1} \ p^r \ q^{x-r} = p^r \sum_{x=r}^{\infty} C_{r-1}^{x-1} \ q^{x-r}$$

$$= p^r \left(1 + rq + \frac{r(r+1)}{2!} q^2 + \frac{r(r+1)(r+2)}{3!} q^3 + \cdots \cdots \right)$$

Recall that $\sum_{j=0}^{\infty} C_j^{r+j-1} \ x^j = (1-x)^{-r}$

$$= p^r (1-q)^{-r} = p^r (1 - 1 + p)^{-r} = p^r p^{-r} = 1$$

$$\therefore \sum_{x=r}^{\infty} C_{r-1}^{x-1} \ p^r \ q^{x-r} = 1$$

3. The **mean** of the r.v. $X \sim NB(r, p)$ can be obtain as follows

$$\mu_x = E(x) = \frac{r}{p}$$

4. The **variance** of the r.v. $X \sim NB(r,p)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{rq}{p^2}$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \le x) = \begin{cases} 0 & x < r \\ \sum_{k=r}^{x} C_{r-1}^{k-1}\ p^r\ q^{k-r} & r \le x < \infty \\ 1 & x \to \infty \end{cases}$$

Or

$$F(x) = \begin{cases} 0 & x < r \\ \sum_{k=r}^{x} C_{r-1}^{k-1}\ p^r\ q^{k-r} & x \ge r \end{cases}$$

<u>Ex.16:</u> suppose we flip a fair coin until we get four heads. Let the r.v. X denote the number of trials to get four head.

1. Find the probability mass function of X.
2. Find the cumulative distribution function of X.
3. Find the mean (μ_x) and the variance (σ_x^2).

Solution:

1. $X \sim NB(r,p)$ $\quad ,r = 4,\ p = 0.5,\ q = 1 - p = 1 - 0.5 = 0.5$

$$p(x,r,p) = \begin{cases} C_{r-1}^{x-1}\ p^r\ q^{x-r} & x = r, r+1, r+2, \dots \dots \\ 0 & otherwise \end{cases}$$

$$p(x,r,p) = \begin{cases} C_3^{x-1}\ (0.5)^4\ (0.5)^{x-4} & x = 4,5,6, \dots \dots \\ 0 & otherwise \end{cases}$$

2.

$$F(x) = \begin{cases} 0 & x < r \\ \sum_{k=r}^{x} C_{r-1}^{k-1}\ p^r\ q^{k-r} & x \ge r \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 4 \\ \sum_{k=4}^{x} C_3^{k-1} \ (0.5)^4 \ (0.5)^{k-4} & x \geq 4 \end{cases}$$

3. $\mu_x = E(x) = \frac{r}{p} = \frac{4}{0.5} = 8$, $var(x) = \sigma_x^2 = \frac{rq}{p^2} = \frac{4*0.5}{(0.5)^2} = \frac{2}{0.25} = 8$

<u>Ex.17:</u> A fair die is rolled until the number two fours occur. Let the random variable X denote the number of failures before obtaining two fours.

1. Write the probability mass function of X.
2. Find the cumulative distribution function of X.
3. Find the mean (μ_x) and the variance (σ_x^2).
4. Find the probability that exactly 5 tosses are needed.
5. Find the probability that at most 5 tosses are needed.

Solution:

1. $X \sim NB(r, p)$ $, r = 2,\ p = \frac{1}{6},\ \ q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

$$p(x, r, p) = \begin{cases} C_x^{x+r-1} \ p^r \ q^x & x = 0,1,2, \dots \dots \\ 0 & otherwise \end{cases}$$

$$p(x, r, p) = \begin{cases} C_x^{x+2-1} \ \left(\frac{1}{6}\right)^2 \ \left(\frac{5}{6}\right)^x & x = 0,1,2, \dots \dots \\ 0 & otherwise \end{cases}$$

2. $F(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} C_k^{k+r-1} \ p^r \ q^k & x \geq 0 \end{cases}$

$$F(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} C_k^{k+2-1} \ \left(\frac{1}{6}\right)^2 \ \left(\frac{5}{6}\right)^k & x \geq 0 \end{cases}$$

3. $\mu_x = E(x) = \frac{rq}{p} = \frac{2*(5/6)}{1/6} = 10$, $var(x) = \sigma_x^2 = \frac{rq}{p^2} = \frac{2*(5/6)}{(1/6)^2} = 60$

4. $P(x = 5) = C_5^{5+2-1} \ \left(\frac{1}{6}\right)^2 \ \left(\frac{5}{6}\right)^5 = 0.0669$

5. $P(x \leq 5) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5)$

$$= C_0^{0+2-1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^0 + C_1^{1+2-1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^1 + C_2^{2+2-1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 + C_3^{3+2-1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 + C_4^{4+2-1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4 + C_5^{5+2-1} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^5 = 0.330$$

7. Hyper geometric Distribution

Suppose we have a population of N objects, D of one type and N-D of a second type. A random sample of size n is drawn from the population without replacement. Let the r.v. X denote the number of objects of the first type selected. Then the probability mass function of X is:

$$p(x,n,N,D)=\begin{cases}\dfrac{C_x^D\ C_{n-x}^{N-D}}{C_n^N} & x=a,a+1,a+2,\ldots\ldots,b\\ 0 & otherwise\end{cases}$$

$$a=max\{0,n-N+D\}\ ,\qquad b=min\{n,D\}$$

1. $0\le p(x,n,N,D)\le 1$
2. $\sum_{\forall x}p(x,n,N,D)=1$

$$\sum_{x=a}^{b}p(x,n,N,D)=\sum_{x=a}^{b}\frac{C_x^D\ C_{n-x}^{N-D}}{C_n^N}=\frac{1}{C_n^N}\sum_{x=a}^{b}C_x^D\ C_{n-x}^{N-D}$$

Recall that $\sum_{x=a}^{b}C_x^D\ C_{n-x}^{N-D}=C_n^N$

$$=\frac{C_n^N}{C_n^N}=1$$

$$\therefore\sum_{x=a}^{b}\frac{C_x^D\ C_{n-x}^{N-D}}{C_n^N}=1$$

3. The **mean** of the r.v. $X\sim HG(n,N,D)$ can be obtain as follows

$$\mu_x=E(x)=\frac{nD}{N}$$

4. The **variance** of the r.v. $X\sim HG(n,N,D)$ can be obtain as follows

$$var(x)=\sigma_x^2=E(x^2)-(E(x))^2=\frac{nD}{N}\frac{N-D}{N}\frac{N-n}{N-1}$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} \frac{C_k^D \; C_{n-k}^{N-D}}{C_n^N} & 0 \le x < mim(n,D) \\ 1 & x \ge mim(n,D) \end{cases}$$

Ex.18: A lot consists of 100 fuses, of which 20 are defective. A random sample of five fuses is selected without replacement. Let the random variable X denote the number of defective fuses in the sample.

1. Write the probability mass function of X.
2. Find the cumulative distribution function of X.
3. Find the mean (μ_x) and the variance (σ_x^2).
4. Find the probability that there is no defective fuse in the sample.
5. Find the probability that there is one defective fuse in the sample

Solution:

1. $X \sim HG(n, N, D) \quad , N = 100, \; n = 5, \; D = 20$

$$a = max\{0, n - N + D\} = max\{0, 5 - 100 + 20\}$$
$$= max\{0, 5 - 100 + 20\} = max\{0, -75\} = 0$$

$$b = min\{n, D\} = min\{5, 20\} = 5$$

$$p(x, n, N, D) = \begin{cases} \frac{C_x^D \; C_{n-x}^{N-D}}{C_n^N} & x = a, a+1, a+2, \dots\dots, b \\ 0 & otherwise \end{cases}$$

$$p(x, n, N, D) = \begin{cases} \frac{C_x^{20} \; C_{5-x}^{80}}{C_5^{100}} & x = 0,1,2,3,4,5 \\ 0 & otherwise \end{cases}$$

2.

$$F(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} \frac{C_k^D \; C_{n-k}^{N-D}}{C_n^N} & 0 \le x < mim(n,D) \\ 1 & x \ge mim(n,D) \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{x} \frac{C_k^{20} \; C_{5-k}^{80}}{C_5^{100}} & 0 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

3. $\mu_x = E(x) = \frac{nD}{N} = \frac{5*20}{100} = 1$

$$var(x) = \sigma_x^2 = \frac{nD}{N} \frac{N-D}{N} \frac{N-n}{N-1} = \frac{5*20}{100} * \frac{100-20}{100} * \frac{100-5}{100-1}$$
$$= 0.768$$

4. $P(x = 0) = \frac{C_x^{20} \; C_{5-x}^{80}}{C_5^{100}} = \frac{C_0^{20} \; C_{5-0}^{80}}{C_5^{100}} = 0.3193$

5. $P(x = 1) = \frac{C_x^{20} \; C_{5-x}^{80}}{C_5^{100}} = \frac{C_1^{20} \; C_{5-1}^{80}}{C_5^{100}} = 0.420$