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قسم الاحصاء

التوزيعات الاحتمالية

Probability Distributions

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Chapter Three

Some Continuous Probability Distributions

1. Continuous Uniform Distribution

A random variable X is said to have a continuous uniform distribution on the interval (a, b) if its probability density function (p.d.f.) is given by:

$$f(x, a, b) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & o.w. \end{cases}$$

Where the parameters a, b satisfy $-\infty < a < b < \infty$, we will write $X \sim U(a, b)$.

1. $f(x, a, b) \geq 0$
2. $\int_{\forall x} f(x, a, b)dx = 1$

$$\int_a^b \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b dx = \frac{1}{b-a} \, [x]_a^b = \frac{b-a}{b-a} = 1$$

3. The **mean** of the r.v. $X \sim U(a, b)$ can be obtain as follows

$$\mu_x = E(x) = \frac{a+b}{2}$$

4. The **variance** of the r.v. $X \sim U(a, b)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{(b-a)^2}{12}$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_a^x \frac{1}{b-a} dt = \frac{1}{b-a} \int_a^x dt$$

$$= \frac{1}{b-a} \, [t]_a^x = \frac{x-a}{b-a}$$

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & x \ge b \end{cases}$$

<u>Ex.1</u>: If $X \sim U(0,10)$.

1. Write the probability density function of X.
2. $P(x < 3), P(x > 6), and\ P(3 < x < 8)$ using the p.d.f.
3. What is the cumulative distribution function of X.
4. $P(x < 3), P(x > 6), and\ P(3 < x < 8)$ using the c.d.f.
5. Find the mean and the variance of X.

Solution:

1. $f(x) = \begin{cases} \frac{1}{10} & 0 \le x \le 10 \\ 0 & o.w. \end{cases}$

2. $P(x < 3) = \int_0^3 f(x)dx = \int_0^3 \frac{1}{10}\, dx = \frac{1}{10}\ [x]_0^3 = \frac{3}{10}$

$P(x > 6) = \int_6^{10} f(x)dx = \int_6^{10} \frac{1}{10}\, dx = \frac{1}{10}\ [x]_6^{10} = \frac{10-6}{10} = \frac{4}{10}$

$P(3 < x < 8) = \int_3^8 f(x)dx = \int_3^{10} \frac{1}{10}\, dx = \frac{1}{10}\ [x]_3^8 = \frac{8-3}{10} = \frac{5}{10}$

3. $F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & x \ge b \end{cases}$

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{10} & 0 \le x < 10 \\ 1 & x \ge 10 \end{cases}$$

4. We can use $F(x)$ to compute the probabilities above.

$$P(x < 3) = F(3) - F(0) = \frac{3}{10} - 0 = \frac{3}{10}$$

$$P(x > 6) = F(10) - F(6) = 1 - \frac{6}{10} = \frac{4}{10}$$

$$P(3 < x < 8) = F(8) - F(3) = \frac{8}{10} - \frac{3}{10} = \frac{5}{10}$$

5. $\mu_x = \frac{a+b}{2} = \frac{0+10}{2} = 5$

$$var(x) = \sigma_x^2 = \frac{(b-a)^2}{12} = \frac{(10-0)^2}{12} = \frac{100}{12}$$

<u>Ex.2</u>: A bus arrives at a station at a random time between 8:10 AM and 8:30 AM. Assume that the arrival time is uniformly distributed over this interval.

1. Write the probability density function of X.
2. Find the probability that the bus arrives before 8:15 AM.
3. Find the probability that the bus arrives after 8:20 AM.
4. Find $P(13 < x < 18)$.
5. What is the cumulative distribution function of X.
6. Find the mean and the variance of X.

Solution:

1. $X \sim U(10,30)$

$$f(x) = \begin{cases} \frac{1}{20} & 10 \le x \le 30 \\ 0 & o.w. \end{cases}$$

2. $P(x < 15) = \int_{10}^{15} f(x)dx = \int_{10}^{15} \frac{1}{20}\, dx = \frac{1}{20}\, [x]_{10}^{15} = \frac{15-10}{20} = \frac{5}{20}$

3. $P(x > 20) = \int_{20}^{30} f(x)dx = \int_{20}^{30} \frac{1}{20}\, dx = \frac{1}{20}\, [x]_{20}^{30} = \frac{30-20}{20} = \frac{10}{20}$

4. $P(13 < x < 18) = \int_{13}^{18} f(x)dx = \int_{13}^{18} \frac{1}{20}\, dx = \frac{1}{20}\, [x]_{13}^{18} = \frac{18-13}{20} = \frac{5}{20}$

5. $F(x) = \begin{cases} 0 & x < 10 \\ \frac{x-10}{20} & 10 \le x < 30 \\ 1 & x \ge 30 \end{cases}$

6. $\mu_x = \frac{a+b}{2} = \frac{10+30}{2} = \frac{40}{2} = 20$

$$var(x) = \sigma_x^2 = \frac{(b-a)^2}{12} = \frac{(30-10)^2}{12} = \frac{400}{12}$$

2. The Normal Distribution

A random variable X is said to have a normal distribution with parameters μ and σ^2 if its probability density function (p.d.f.) is given by:

$$f(x,\mu,\sigma^2) = \begin{cases} \frac{1}{\sqrt{2\,\pi\,\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\,\sigma^2}} & -\infty < x < \infty \\ 0 & o.w. \end{cases}$$

Where $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$, the parameter μ represents the mean of x and the parameter σ^2 represents the variance of x. We write $X \sim N(\mu, \sigma^2)$.

1. $f(x,\mu,\sigma^2) \geq 0$
2. $\int_{\forall x} f(x,\mu,\sigma^2)dx = 1$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\,\pi\,\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\,\sigma^2}}\, dx = \frac{1}{\sqrt{2\,\pi\,\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\,\sigma^2}}\, dx$$

Let $z = \frac{x-\mu}{\sigma} \Longrightarrow z\sigma = x - \mu$

$\sigma\, dz = dx$

$$\frac{1}{\sqrt{2\,\pi\,\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\ \sigma\, dz$$

$$\frac{\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\ dz$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\ dz$$

Recall that: $\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\ dz = \sqrt{2\pi}$

$$\frac{1}{\sqrt{2\pi}} \sqrt{2\pi} = 1$$

3. The **mean** of the r.v. $X \sim N(\mu, \sigma^2)$ can be obtain as follows

$$\mu_x = E(x) = \mu$$

4. The **variance** of the r.v. $X \sim N(\mu, \sigma^2)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \sigma^2$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\,\pi\,\sigma^2}}\, e^{-\frac{(t-\mu)^2}{2\,\sigma^2}}\, dt$$

Remark:

1. If $X \sim N(\mu, \sigma^2)$, then $z = \frac{x-\mu}{\sigma}$ is a **Standard Normal Distribution** with mean 0 and variance 1 $z \sim SN(0,1)$.
2. The probability density function (p.d.f.) of the Standard Normal Distribution is given by:

$$f(z) = \begin{cases} \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{z^2}{2}} & -\infty < z < \infty \\ 0 & other\ wise \end{cases}$$

3. $P(a < z < b) = \phi(b) - \phi(a)$. $\phi(b), \phi(a)$ are calculate from the tables.
4. $P(z \le a) = \phi(a)$
5. $P(z > a) = 1 - P(z \le a) = 1 - \phi(a)$
6. $P(z < -a) = P(z > a) = 1 - P(z \le a) = 1 - \phi(a)$
7. $\phi(-a) = 1 - \phi(a)$

Ex.3: Let $X \sim N(4{,}25)$. $\phi(0.4) = 0.6554$, $\phi(0.2) = 0.5793$

1. Write the p.d.f. of X.
2. Find the mean and the variance of X.
3. Find $P(x < 6)$.
4. Find $P(3 < x < 5)$.
5. Find $P(x \le 2)$.

Solution:

1. $X \sim N(4,25)$

$$f(x,\mu,\sigma^2) = \begin{cases} \frac{1}{\sqrt{2\,\pi\,25}}\, e^{-\frac{(x-4)^2}{2\,(25)}} & -\infty < x < \infty \\ 0 & o.w. \end{cases}$$

$$f(x,4,25) = \begin{cases} \frac{1}{\sqrt{50\,\pi}}\, e^{-\frac{(x-4)^2}{50}} & -\infty < x < \infty \\ 0 & o.w. \end{cases}$$

2. $\mu_x = E(x) = \mu = 4\,,\qquad var(x) = \sigma_x^2 = \sigma^2 = 25$

3. $P(x < 6) = P\left(\frac{x-\mu}{\sigma} < \frac{6-\mu}{\sigma}\right) = P\left(z < \frac{6-4}{5}\right) = P\left(z < \frac{2}{5}\right)$

$$= P(z < 0.4) = \phi(0.4) = 0.6554$$

4. $P(3 < x < 5) = P\left(\frac{3-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{5-\mu}{\sigma}\right) = P\left(\frac{3-4}{5} < z < \frac{5-4}{5}\right)$

$$= P(-0.2 < z < 0.2) = \phi(0.2) - \phi(-0.2)$$

$$= \phi(0.2) - [1 - \phi(0.2)] = 0.5793 - [1 - 0.5793] = 0.1586$$

5. $P(x \le 2) = P\left(\frac{x-\mu}{\sigma} \le \frac{2-\mu}{\sigma}\right) = P\left(z < \frac{2-4}{5}\right) = P\left(z < \frac{-2}{5}\right)$

$$= P(z < -0.4) = P(z > 0.4) = 1 - P(z \le 0.4)$$

$$= 1 - \phi(0.4) = 1 - 0.6554 = 0.3446$$

Or

$$P(x \le 2) = P\left(\frac{x-\mu}{\sigma} \le \frac{2-\mu}{\sigma}\right) = P\left(z < \frac{2-4}{5}\right) = P\left(z < \frac{-2}{5}\right)$$

$$= P(z < -0.4) = \phi(-0.4)$$

$$= 1 - \phi(0.4) = 1 - 0.6554 = 0.3446$$

Ex.4: Suppose the temperature in Iraq during July follows a normal distribution with mean $45\ ℃$ and standard deviation 2.5 ℃. Find the probability that the temperature is:

1. Between $46.7\ ℃$ *and* $48.1\ ℃$.
2. Greater than $47.5\ ℃$.

Solution:

$X \sim N(45,(2.5)^2) \implies X \sim N(45{,}6.25)$

$$f(x,\mu,\sigma^2) = \begin{cases} \dfrac{1}{\sqrt{12.5\,\pi}}\, e^{-\frac{(x-45)^2}{12.5}} & -\infty < x < \infty \\ 0 & o.w. \end{cases}$$

1. $P(46.7 < x < 48.1) = P\left(\frac{46.7-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{48.1-\mu}{\sigma}\right)$

$$= P\left(\frac{46.7-45}{2.5} < z < \frac{48.1-45}{2.5}\right)$$

$$= P(0.68 < z < 1.24)$$

$$= \phi(1.24) - \phi(0.68) = 0.8925 - 0.7517 = 0.1408$$

2. $P(x > 47.5) = P\left(\frac{x-\mu}{\sigma} > \frac{47.5-\mu}{\sigma}\right) = P\left(z > \frac{47.5-45}{2.5}\right)$

$$= P(z > 1) = 1 - P(z < 1)$$

$$= 1 - \phi(1) = 1 - 0.8413 = 0.1587$$

Ex.5: Let $Z \sim SN(0{,}1)$.

1. Find $P(z < 1.5)$.
2. Find $P(z > 2)$.
3. Find $P(z \le -1.2)$.
4. Find $P(-0.8 < z < 0.6)$.

Solution:

$Z \sim N(0{,}1)$

$$f(z) = \begin{cases} \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{z^2}{2}} & -\infty < x < \infty \\ 0 & o.w. \end{cases}$$

1. $P(z < 1.5) = \phi(1.5) = 0.9332$

2. $P(z > 2) = 1 - P(z \leq 2) = 1 - \phi(2) = 1 - 0.9772 = 0.0228$

3. $P(z \leq -1.2) = \phi(-1.2) = 1 - \phi(1.2) = 1 - 0.8849 = 0.1151$

4. $P(-0.8 < z < 0.6) = \phi(0.6) - \phi(-0.8) = \phi(0.6) - [1 - \phi(0.8)]$

$$= 0.7275 - [1 - 0.7881] = 0.5138$$

3. The Exponential Distribution

The exponential distribution is one of the important continuous distributions, commonly used in modeling operating times, survival times and waiting times. A random variable X is said to have an exponential distribution if its probability density function (p.d.f.) is given by:

$$f(x,\theta) = \begin{cases} \theta\ e^{-\theta x} & x > 0 \\ 0 & o.w. \end{cases}$$

Where $\theta > 0$.

1. $f(x,\theta) \geq 0$
2. $\int_{\forall x} f(x,\theta)dx = 1$

   $$\int_0^{\infty} \theta\ e^{-\theta x}\ dx = -\int_0^{\infty} -\theta\ e^{-\theta x}\ dx$$

   $$= -\left[e^{-\theta x}\right]_0^{\infty} = -[e^{-\infty} - e^{0}] = 1$$

3. The **mean** of the r.v. $X \sim exp(\theta)$ can be obtain as follows

   $$\mu_x = E(x) = \frac{1}{\theta}$$

4. The **variance** of the r.v. $X \sim exp(\theta)$can be obtain as follows

   $$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{1}{\theta^2}$$

5. The Cumulative Distribution Function of X is given by:

   $$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\ dt = \int_0^{x} \theta\ e^{-\theta t}\ dt$$

   $$= -\int_0^{x} -\theta\ e^{-\theta t}\ dt = -\left[e^{-\theta t}\right]_0^{x} = -\left[e^{-\theta x} - e^{0}\right]$$

   $$= -e^{-\theta x} + 1 = 1 - e^{-\theta x}$$

   $$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\theta x} & 0 \leq x < \infty \end{cases}$$

Remark: Another formula for the exponential distribution is given by:

$$f(x,\theta)=\begin{cases}\frac{1}{\theta}\ e^{-\frac{x}{\theta}} & x>0\\ 0 & o.w.\end{cases}$$

Where $\theta>0$.

1. $f(x,\theta)\geq 0$
2. $\int_{\forall x} f(x,\theta)dx=1$

   $\int_0^\infty \frac{1}{\theta}\ e^{-\frac{x}{\theta}}\ dx=-\ \int_0^\infty -\ \frac{1}{\theta}\ e^{-\frac{x}{\theta}}\ dx$

   $$=-\left[e^{-\frac{x}{\theta}}\right]_0^\infty=-[e^{-\infty}-e^0]=1$$

3. The **mean** of the r.v. $X\sim exp(\theta)$ can be obtain as follows

   $$\mu_x=E(x)=\theta$$

4. The **variance** of the r.v. $X\sim exp(\theta)$can be obtain as follows

   $$var(x)=\sigma_x^2=E(x^2)-(E(x))^2=\theta^2$$

5. The Cumulative Distribution Function of X is given by:

$$F(x)=P(X\leq x)=\int_{-\infty}^{x} f(t)\ dt=\int_0^x \frac{1}{\theta}\ e^{-\frac{t}{\theta}}\ dt$$

$$=-\int_0^x -\frac{1}{\theta}\ e^{-\frac{t}{\theta}}\,dt=-\left[e^{-\frac{t}{\theta}}\right]_0^x=-\left[e^{-\frac{x}{\theta}}-e^0\right]$$

$$=-e^{-\frac{x}{\theta}}+1=1-e^{-\frac{x}{\theta}}$$

$$F(x)=\begin{cases}0 & x<0\\ 1-e^{-\frac{x}{\theta}} & 0\leq x<\infty\end{cases}$$

Ex.6: Suppose that the length of a phone call (in minutes) is an exponential random variable with parameter $\theta = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait,

1. More than 10 minutes.
2. Between 10 and 20 minutes.

Solution:

$$X \sim exp\left(\theta = \frac{1}{10}\right)$$

$$f(x,\theta) = \begin{cases} \frac{1}{10}\, e^{-\frac{x}{10}} & x > 0 \\ 0 & o.w. \end{cases}$$

1. $P(x > 10) = \int_{10}^{\infty} \frac{1}{10}\, e^{-\frac{x}{10}}\, dx = -\int_{10}^{\infty} -\frac{1}{10}\, e^{-\frac{x}{10}}\, dx$

$$= -\left[e^{-\frac{x}{10}}\right]_{10}^{\infty} = -\left[e^{-\infty} - e^{-10/10}\right] = -\left[\frac{1}{e^{\infty}} - e^{-1}\right]$$

$$= -[0 - e^{-1}] = e^{-1} = 0.368$$

2. $P(10 < x < 20) = \int_{10}^{20} \frac{1}{10}\, e^{-\frac{x}{10}}\, dx = -\int_{10}^{20} -\frac{1}{10}\, e^{-\frac{x}{10}}\, dx$

$$= -\left[e^{-\frac{x}{10}}\right]_{10}^{20} = -\left[e^{-20/10} - e^{-10/10}\right] = -[e^{-2} - e^{-1}]$$

$$= e^{-1} - e^{-2} = 0.233$$

Ex.7: Let $X \sim exp(2)$.

1. Write the p.d.f. of X.
2. Find the mean and the variance of X.
3. Find the cumulative distribution function of X.
4. $P(x > 2), P(3 < x < 5), P(|x| < 3)$.
5. $P(2 < x \leq 4)$ using c.d.f.

Solution:

1. $X \sim exp(2)$

$$f(x,2) = \begin{cases} 2\, e^{-2x} & x > 0 \\ 0 & o.w. \end{cases}$$

2. $\mu_x = E(x) = \frac{1}{\theta} = \frac{1}{2}$, $\quad var(x) = \sigma_x^2 = \frac{1}{\theta^2} = \frac{1}{4}$

3.

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-2x} & 0 \le x < \infty \end{cases}$$

4.

$$\Longrightarrow P(x > 2) = \int_2^\infty 2\, e^{-2x}\, dx = -\int_2^\infty -2\, e^{-2x}\, dx$$

$$= -[e^{-2x}]_2^\infty = -[e^{-\infty} - e^{-4}] = e^{-4} = 0.0183$$

$$\Longrightarrow P(3 < x < 5) = \int_3^5 2\, e^{-2x}\, dx = -\int_3^5 -2\, e^{-2x}\, dx$$

$$= -[e^{-2x}]_3^5 = -[e^{-10} - e^{-6}] = e^{-6} - e^{-10} = 0.00243$$

$$\Longrightarrow P(|x| < 3) = P(-3 < x < 3) = P(-3 < x < 0) + P(0 < x < 3)$$
$$= 0 + P(0 < x < 3) = P(0 < x < 3)$$

$$= \int_0^3 2\, e^{-2x}\, dx = -\int_0^3 -2\, e^{-2x}\, dx = -[e^{-2x}]_0^3 = -[e^{-3} - e^0]$$
$$= 1 - e^{-3} = 0.95$$

5. $P(2 < x \le 4) = F(4) - F(2)$

$$= (1 - e^{-8}) - (1 - e^{-4}) = e^{-4} - e^{-8} = 0.01797$$

Ex.8: Let the r.v. X denote the operating times (in hours) of a machine, if $X \sim exp(0.04)$. Answer each of the following:

1. Write the p.d.f. of X.
2. Write the c.d.f. of X.
3. Find the mean of machine operating times.
4. Find the probability that the machine will work at least 20 hours.

Solution:

1. $X \sim exp(0.04)$

$$f(x, 0.04) = \begin{cases} 0.04\, e^{-0.04x} & x > 0 \\ 0 & o.w. \end{cases}$$

2.

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-0.04\,x} & 0 \le x < \infty \end{cases}$$

3. $\mu_x = \frac{1}{\theta} = \frac{1}{0.04} = 25\, hours$

4. $P(x \ge 20) = \int_{20}^{\infty} 0.04\, e^{-0.04x}\, dx = -\int_{20}^{\infty} -0.04\, e^{-0.04x}\, dx$

$$= -[e^{-0.04x}]_{20}^{\infty} = -[e^{-\infty} - e^{-0.8}] = e^{-0.8} = 0.449$$

<u>Ex.9</u>: Let the r.v. X denote the operating times (in hours) of a machine, suppose the average operating time of the machine is 25 hours. Answer each of the following:

1. Write the p.d.f. of X.
2. Write the c.d.f. of X.
3. Find the mean of machine operating times.
4. Find the probability that the machine will work at least 20 hours.

Solution:

1. $X \sim exp(25)$

$$f(x, 0.04) = \begin{cases} \frac{1}{25}\, e^{-x/25} & x > 0 \\ 0 & o.w. \end{cases}$$

2.

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/25} & 0 \le x < \infty \end{cases}$$

3. $\mu_x = \theta = 25\, hours$

4. $P(x \ge 20) = \int_{20}^{\infty} \frac{1}{25}\, e^{-x/25}\, dx = -\int_{20}^{\infty} -\frac{1}{25}\, e^{-x/25}\, dx$

$$= -\left[e^{-x/25}\right]_{20}^{\infty} = -[e^{-\infty} - e^{-0.8}] = e^{-0.8} = 0.449$$

4. Gamma Distribution

A random variable X is said to have a gamma distribution with parameters $\alpha \; and \; \beta$ if its probability density function (p.d.f.) is given by:

$$f(x,\alpha,\beta) = \begin{cases} \frac{\beta^{\alpha}}{\Gamma\alpha} \; x^{\alpha-1} \, e^{-\beta x} & x > 0 \\ 0 & o.w. \end{cases}$$

$$\Gamma\alpha = \int_{0}^{\infty} x^{\alpha-1} \, e^{-x} \; dx = (\alpha-1)! \qquad \textit{Gamma Function}$$

Where the parameters $\alpha, \beta > 0$.

1. $f(x,\alpha,\beta) \geq 0$
2. $\int_{\forall x} f(x,\alpha,\beta)dx = 1$

$$\int_0^{\infty} \frac{\beta^{\alpha}}{\Gamma\alpha} \; x^{\alpha-1} \, e^{-\beta x} \, dx = \frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^{\infty} x^{\alpha-1} \, e^{-\beta x} \, dx$$

Let $y = \beta x \Longrightarrow x = \frac{y}{\beta} \Longrightarrow dx = \frac{1}{\beta} \, dy$

Substitute

$$\frac{\beta^{\alpha}}{\Gamma\alpha} \int_0^{\infty} \left(\frac{y}{\beta}\right)^{\alpha-1} \; e^{-y} \, \frac{1}{\beta} \, dy$$

$$\frac{\beta^{\alpha}}{\Gamma\alpha} * \frac{1}{\beta^{\alpha}} \int_{0}^{\infty} y^{\alpha-1} \, e^{-y} \; dy$$

$$\frac{1}{\Gamma\alpha} \int_{0}^{\infty} y^{\alpha-1} \, e^{-y} \; dy$$

$$\Gamma\alpha = \int_{0}^{\infty} y^{\alpha-1} \, e^{-y} \; dy$$

$$\frac{1}{\Gamma\alpha} * \Gamma\alpha = 1$$

3. The **mean** of the r.v. $X \sim Gamm(\alpha,\beta)$ can be obtain as follows

$$\mu_x = E(x) = \frac{\alpha}{\beta}$$

4. The **variance** of the r.v. $X \sim Gamm(\alpha,\beta)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{\alpha}{\beta^2}$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\ dt = \int_0^x \frac{\beta^\alpha}{\Gamma\alpha}\ t^{\alpha-1}\ e^{-\beta t}\ dt$$

Remark 1: Another formula for the Gamma distribution is given by:

$$f(x,\alpha,\beta) = \begin{cases} \dfrac{1}{\Gamma\alpha\ \beta^\alpha}\ x^{\alpha-1}\ e^{-x/\beta} & x > 0 \\ 0 & o.w. \end{cases}$$

$$\Gamma\alpha = \int_0^\infty x^{\alpha-1}\ e^{-x}\ dx = (\alpha-1)! \qquad Gamma\ Function$$

Where the parameters $\alpha,\beta > 0$.

1. $f(x,\alpha,\beta) \ge 0$
2. $\int_{\forall x} f(x,\alpha,\beta)dx = 1$

$$\int_0^\infty \frac{1}{\Gamma\alpha\ \beta^\alpha}\ x^{\alpha-1}\ e^{-x/\beta}\ dx = \frac{1}{\Gamma\alpha\ \beta^\alpha}\int_0^\infty x^{\alpha-1}\ e^{-x/\beta}\ dx$$

Let $y = \frac{x}{\beta} \Rightarrow x = \beta y \Rightarrow dx = \beta\ dy$

Substitute

$$\frac{1}{\Gamma\alpha\ \beta^\alpha}\int_0^\infty (\beta y)^{\alpha-1}\ e^{-y}\ \beta\ dy$$

$$\frac{\beta^\alpha}{\Gamma\alpha\ \beta^\alpha}\int_0^\infty y^{\alpha-1}\ e^{-y}\ dy$$

$$\frac{1}{\Gamma\alpha}\int_{0}^{\infty} y^{\alpha-1}\, e^{-y}\; dy$$

$$\Gamma\alpha = \int_{0}^{\infty} y^{\alpha-1}\, e^{-y}\; dy$$

$$\frac{1}{\Gamma\alpha} * \Gamma\alpha = 1$$

3. The **mean** of the r.v. $X \sim Gamm(\alpha, \beta)$ can be obtain as follows

$$\mu_x = E(x) = \alpha\beta$$

4. The **variance** of the r.v. $X \sim Gamm(\alpha, \beta)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \alpha\beta^2$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\; dt = \int_{0}^{x} \frac{1}{\Gamma\alpha\, \beta^{\alpha}}\; t^{\alpha-1}\, e^{-t/\beta}\; dt$$

Remark 2: If $\alpha = 1$, X is said to have an exponential distribution with parameter β, $X \sim exp(\beta)$

$$f(x, \alpha, \beta) = \frac{\beta^1}{\Gamma 1}\; x^{1-1}\, e^{-\beta x}$$

$$f(x, \beta) = \begin{cases} \beta\;\; e^{-\beta x} & x > 0 \\ \;\; 0 & o.w. \end{cases}$$

Remark 3: If $= \frac{n}{2}$ $and\ \beta = \frac{1}{2}$, X is said to have an chi-square distribution with parameter n, $X \sim \chi^2(n)$

$$f(x, \alpha, \beta) = \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma\frac{n}{2}}\; x^{\frac{n}{2}-1}\, e^{-\frac{1}{2}x}$$

$$f(x,n) = \begin{cases} \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \; x^{\frac{n}{2}-1} \, e^{-\frac{1}{2}x} & x > 0 \\ 0 & o.w. \end{cases}$$

<u>Ex.10</u>: Let $X \sim Gamm(2,4)$

1. Write the p.d.f. of X.
2. Find $P(x < 3)$.

Solution:

1. $f(x,\alpha,\beta) = \begin{cases} \frac{\beta^{\alpha}}{\Gamma\alpha} \; x^{\alpha-1} \, e^{-\beta x} & x > 0 \\ 0 & o.w. \end{cases}$

$$f(x,\alpha,\beta) = \frac{4^2}{\Gamma 2} \; x^{2-1} \, e^{-4x}$$

$$\Gamma 2 = (2-1)! = 1$$

$$f(x,\alpha,\beta) = \begin{cases} 16\, x\, e^{-4x} & x > 0 \\ 0 & o.w. \end{cases}$$

2. $P(x < 3) = \int_0^3 16\, x\, e^{-4x}\, dx = 16 \int_0^3 \; x\, e^{-4x} \; dx$

$$\int u\, dv = uv - \int v\, du$$

$$u = x \quad , \quad du = dx$$

$$dv = e^{-4x} \quad , \quad v = \frac{-1}{4} e^{-4x}$$

$$\int_0^3 x\, e^{-4x} \; dx = \left[x \; \frac{-1}{4} e^{-4x}\right]_0^3 + \int_0^3 \frac{-1}{4} e^{-4x} \; dx$$

$$= \left[\frac{-3}{4} \; e^{-12} - 0\right] - \frac{1}{4} * \frac{1}{4} \int_0^3 (-4) e^{-4x} \; dx$$

$$= \frac{-3}{4} \; e^{-12} - \frac{1}{16} [e^{-4x}]_0^3$$

$$= \frac{-3}{4} \; e^{-12} - \frac{1}{16} [\, e^{-12} - e^{0}]$$

$$= \frac{-3}{4}\ e^{-12} - \frac{1}{16}[\ e^{-12} - e^{0}] = 0.062$$

$$P(x < 3) = 16\int_{0}^{3} x\ e^{-4x}\ dx = 16 * 0.062 = 0.992$$

<u>Ex.11</u>: Let X be a random variable with the following p.d.f.

$$f(x,\alpha,\beta) = \begin{cases} A\ \ x^2\ e^{-5x} & x > 0 \\ 0 & o.w. \end{cases}$$

Find the value of A, what is the name of the distribution.

Solution:

$$f(x,\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma\alpha}\ \ x^{\alpha-1}\ e^{-\beta x}$$

$$f(x,\alpha,\beta) = A\ \ x^2\ e^{-5x}$$

$$\beta = 5\ \ , \alpha - 1 = 2 \Longrightarrow\ \alpha = 3$$

$$A = \frac{\beta^{\alpha}}{\Gamma\alpha} = \frac{5^3}{\Gamma 3} = \frac{125}{2!} = \frac{125}{2}$$

$$f(x,\alpha,\beta) = \begin{cases} \frac{125}{2}\ \ x^2\ e^{-5x} & x > 0 \\ 0 & o.w. \end{cases}$$

$\therefore X \sim Gamm(3,5)$

<u>Ex.12</u>: The lifetime (in hours) of a battery follows a Gamma distribution with $\alpha = 3$ and $\beta = 5\ (scale\ parameter)$

1. Write the probability density function of X.
2. Find the mean and the variance of X.

Solution:

1. $f(x,\alpha,\beta) = \frac{1}{\Gamma 3\ 5^3}\ \ x^{3-1}\ e^{-x/5}$

$\Gamma 3 = (3-1)! = 2! = 2$

$$f(x,\alpha,\beta)=\begin{cases}\frac{1}{250}\ x^2\ e^{-x/5} & x>0\\ 0 & o.w.\end{cases}$$

2. $\mu_x=\alpha\beta=3*5=15\,,\quad var(x)=\alpha\beta^2=3*5^2=75$

Ex.13: The time (in hours) until a certain machine fails follows a Gamma distribution with $\alpha=4$ and rate parameter $\beta=2$ per hour.

1. Write the probability density function of X.
2. Find the mean and the variance of X.

Solution:

1. $f(x,\alpha,\beta)=\frac{2^4}{\Gamma 4}\ x^{4-1}\ e^{-2x}$

$\Gamma 4=(4-1)!=3!=6$

$$f(x,\alpha,\beta)=\begin{cases}\frac{16}{6}\ x^3\ e^{-2x} & x>0\\ 0 & o.w.\end{cases}$$

2. $\mu_x=\frac{\alpha}{\beta}=\frac{4}{2}=2\,,\quad var(x)=\frac{\alpha}{\beta^2}=\frac{4}{2^2}=1$

5. Beta Distribution

A random variable X is said to have a Beta distribution with parameters $a \; and \; b$ if its probability density function (p.d.f.) is given by:

$$f(x,a,b) = \begin{cases} \frac{1}{\mathrm{B(a,b)}} \; x^{a-1} \, (1-x)^{b-1} & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

$$\mathrm{B(a,b)} = \int_0^1 x^{a-1} \, (1-x)^{b-1} \; dx = \frac{\Gamma a \, \Gamma b}{\Gamma a + b} \qquad \textit{Beta Function}$$

Where the parameters $a, b > 0$.

1. $f(x,a,b) \geq 0$
2. $\int_{\forall x} f(x,a,b) dx = 1$

$$\int_0^1 \frac{1}{\mathrm{B(a,b)}} \; x^{a-1} \, (1-x)^{b-1} \, dx$$

$$\frac{1}{\mathrm{B(a,b)}} \int_0^1 x^{a-1} \, (1-x)^{b-1} \, dx$$

$$\frac{1}{\mathrm{B(a,b)}} \, \mathrm{B(a,b)} = 1$$

3. The **mean** of the r.v. $X \sim Beta \; (a,b)$ can be obtain as follows

$$\mu_x = E(x) = \frac{a}{a+b}$$

4. The **variance** of the r.v. $X \sim Beta \; (a,b)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{ab}{(a+b)^2 \, (a+b+1)}$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t) \; dt = \int_0^x \frac{1}{\mathrm{B(a,b)}} \; t^{a-1} \, (1-t)^{b-1} \; dt$$

Ex.14: Let $X \sim Beta(2,3)$

1. Write the p.d.f. of X.
2. Find the c.d.f. of X.
3. Find the mean and the variance of X.
4. Find $P\left(\frac{1}{2} < x < 1\right), P(-1 < x < 1), P(x > 0)$.

Solution:

1. $f(x, a, b) = \frac{1}{\mathrm{B(a,b)}} \; x^{a-1}\,(1-x)^{b-1}$

$$f(x, a, b) = \frac{1}{\mathrm{B}(2{,}3)} \; x^{2-1}\,(1-x)^{3-1}$$

$$\mathrm{B}(2{,}3) = \frac{\Gamma 2\;\Gamma 3}{\Gamma 2 + 3} = \frac{1 * 2}{24} = \frac{1}{12}$$

$$\Gamma 2 = (2-1)! = 1! = 1$$

$$\Gamma 3 = (3-1)! = 2! = 2$$

$$\Gamma 5 = (5-1)! = 4! = 24$$

$$f(x, a, b) = \begin{cases} \frac{1}{1/12} \; x\,(1-x)^2 & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

$$f(x, a, b) = \begin{cases} 12 \; x\,(1-x)^2 & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

2.

$$F(x) = \int_{-\infty}^{x} f(t)\; dt = \int_{0}^{x} 12 \; t\,(1-t)^2 \; dt$$

$$= 12 \int_0^x t\,(1 - 2t + t^2) = 12 \int_0^x t - 2t^2 + t^3$$

$$= 12 \left[\frac{t^2}{2} - 2\frac{t^3}{3} + \frac{t^4}{4}\right]_0^x = 12\left[\left(\frac{x^2}{2} - 2\frac{x^3}{3} + \frac{x^4}{4}\right) - 0\right]$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 6x^2 - 8x^3 + 3x^4 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

3. $\mu_x = \frac{a}{a+b} = \frac{2}{2+3} = \frac{2}{5}$

$$var(x) = \frac{ab}{(a+b)^2\ (a+b+1)} = \frac{2*3}{(2+3)^2\ (2+3+1)} = \frac{6}{150}$$

4. $P\left(\frac{1}{2} < x < 1\right) = F(1) - F\left(\frac{1}{2}\right) = 1 - 0.6875 = 0.3125$

$F(1) = 1$

$$F\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4 = 0.6875$$

$$P(-1 < x < 1) = P(-1 < x < 0) + P(0 < x < 1) = P(0 < x < 1)$$
$$= F(1) - F(0) = 1 - 0 = 1$$

$P(x > 0) = 1 - P(x \le 0) = 1 - F(0) = 1 - 0 = 1$

<u>Ex.15</u>: Let X be a random variable with the following p.d.f.

$$f(x,\alpha,\beta) = \begin{cases} A\ x^3\ (1-x)^2 & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

1. Find the value of A.
2. What is the name of the distribution?
3. Find the c.d.f. of X.
4. Find $P\left(0 < x < \frac{1}{2}\right), P(x < 1), P(\frac{1}{2} < x < 3)$.

Solution:

1. $f(x,a,b) = \frac{1}{\mathrm{B(a,b)}}\ x^{a-1}\ (1-x)^{b-1}$

$f(x,a,b) = A\ x^3\ (1-x)^2$

$a - 1 = 3 \Longrightarrow \boldsymbol{a = 4}\ , b - 1 = 2 \Longrightarrow\ \boldsymbol{b = 3}$

$$A = \frac{1}{\mathrm{B(a,b)}} = \frac{1}{\frac{\Gamma a\, \Gamma b}{\Gamma a + b}} = \frac{\Gamma a + b}{\Gamma a\, \Gamma b} = \frac{\Gamma 4 + 3}{\Gamma 4\, \Gamma 3} = \frac{720}{6*2} = \frac{720}{12} = 60$$

$\Gamma 4 + 3 = \Gamma 7 = (7-1)! = 6! = 720$

$\Gamma 4 = (4-1)! = 3! = 6$

$\Gamma 3 = (3-1)! = 2! = 2$

$$f(x,a,b) = \begin{cases} 60\ x^3\ (1-x)^2 & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

2. $X \sim Beta(4,3)$

3. $F(x) = \int_{-\infty}^{x} f(t)\ dt = \int_0^x 60\ t^3\ (1-t)^2\ dt$

$$= 60 \int_0^x t^3 (1-2t+t^2) = 60 \int_0^x t^3 - 2t^4 + t^5$$

$$= 60 \left[\frac{t^4}{4} - 2\frac{t^5}{5} + \frac{t^6}{6}\right]_0^x = 60\left[\left(\frac{x^4}{4} - 2\frac{x^5}{5} + \frac{x^6}{6}\right) - 0\right]$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 15x^4 - 24x^5 + 10x^6 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

4. $P\left(0 < x < \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F(0) = 0.34375 - 0 = 0.34375$

$$F\left(\frac{1}{2}\right) = 15\left(\frac{1}{2}\right)^4 - 24\left(\frac{1}{2}\right)^5 + 10\left(\frac{1}{2}\right)^6 = 0.34375$$

$$F(0) = 15(0)^4 - 24(0)^5 + 10(0)^6 = 0$$

$$P(x < 1) = F(1) = 15(1)^4 - 24(1)^5 + 10(1)^6 = 1$$

$$P\left(\frac{1}{2} < x < 3\right) = P\left(\frac{1}{2} < x < 1\right) + P(1 < x < 3) = P\left(\frac{1}{2} < x < 1\right)$$

$$= F(1) - F\left(\frac{1}{2}\right) = 1 - 0.34375 = 0.65625$$

6. Weibull Distribution

A random variable X is said to have a Weibull distribution with parameters $\alpha \; and \; \lambda$ if its probability density function (p.d.f.) is given by:

$$f(x,\alpha,\lambda) = \begin{cases} \frac{\alpha}{\lambda} \left(\frac{x}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^{\alpha}} & x \geq 0 \\ 0 & o.w. \end{cases}$$

Where the parameters $\alpha, \lambda > 0$.

1. $f(x,\alpha,\lambda) \geq 0$
2. $\int_{\forall x} f(x,\alpha,\lambda) dx = 1$
3. The **mean** of the r.v. $X \sim Wei\,(\alpha,\lambda)$ can be obtain as follows

$$\mu_x = E(x) = \lambda\,\Gamma\left(1 + \frac{1}{\alpha}\right)$$

4. The **variance** of the r.v. $X \sim Wei\,(\alpha,\lambda)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \lambda^2 \left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \left[\Gamma\left(1 + \frac{1}{\alpha}\right)\right]^2\right]$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\; dt = \int_{0}^{x} \frac{\alpha}{\lambda} \left(\frac{t}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{t}{\lambda}\right)^{\alpha}}\; dt$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\left(\frac{x}{\lambda}\right)^{\alpha}} & x \geq 0 \end{cases}$$

Ex.16: Suppose the life (in hours) of a certain type of light bulb follows a Weibull distribution with parameters $\alpha = 2 \; and \; \lambda = 1000 \; hours.$

($\Gamma 1.5 = 0.8862$,

1. Write the p.d.f. of X.
2. Find the c.d.f. of X.
3. Find the mean and the variance of X.
4. Find the probability that a bulb lasts less than 800 hours.

Solution:

1. $f(x,\alpha,\lambda) = \begin{cases} \frac{2}{1000} \left(\frac{x}{1000}\right)^{2-1} e^{-\left(\frac{x}{1000}\right)^2} & x \geq 0 \\ 0 & o.w. \end{cases}$

2. $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\left(\frac{x}{1000}\right)^2} & x \geq 0 \end{cases}$

3. $\mu_x = \lambda\, \Gamma\left(1 + \frac{1}{\alpha}\right) = 1000\, \Gamma\left(1 + \frac{1}{2}\right) = 1000\, \Gamma\frac{3}{2}$

$$= 1000 * \Gamma 1.5 = 1000 * \ 0.8862 = 886.2\ hours$$

$$var(x) = \lambda^2 \left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \left[\Gamma\left(1 + \frac{1}{\alpha}\right)\right]^2\right]$$

$$= 1000^2 \left[\Gamma\left(1 + \frac{2}{2}\right) - \left[\Gamma\left(1 + \frac{1}{2}\right)\right]^2\right]$$

$$= 1000^2 [\Gamma(2) - [\Gamma(1.5)]^0] = 1000000(1 - (0.8862)^2)$$

$$= 214.700\ hours$$

4. $P(x < 800) = F(800) = 1 - e^{-\left(\frac{800}{1000}\right)^2} = 1 - e^{-0.64}$

$$= 1 - 0.527 = 0.473$$

7. Pareto Distribution

A random variable X is said to have a Pareto distribution with parameter θ if its probability density function (p.d.f.) is given by:

$$f(x,\theta) = \begin{cases} \dfrac{\theta}{x^{\theta+1}} & x \geq 1 \\ 0 & o.w. \end{cases}$$

Where the parameters $\theta > 0$.

1. $f(x,\theta) \geq 0$
2. $\int_{\forall x} f(x,\theta)dx = 1$
3. The **mean** of the r.v. $X \sim Par\ (\theta)$ can be obtain as follows

$$\mu_x = E(x) = \frac{\theta}{\theta - 1}$$

4. The **variance** of the r.v. $X \sim Par\ (\theta)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{\theta}{(\theta-1)^2\,(\theta-2)}$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\ dt = \int_{1}^{x} \frac{\theta}{t^{\theta+1}}\ dt$$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 - \left(\dfrac{1}{x}\right)^{\theta} & x \geq 1 \end{cases}$$

<u>Ex.17</u>: Let X be a random variable that follows Pareto distribution with parameter $\theta = 3$.

1. Write the p.d.f. of X.
2. Find the c.d.f. of X.
3. Find the mean and the variance of X.
4. Find $P(x > 2)$.

Solution:

1. $f(x,\theta) = \begin{cases} \frac{3}{x^4} & x \geq 1 \\ 0 & o.w. \end{cases}$

2. $F(x) = \begin{cases} 0 & x < 1 \\ 1 - \left(\frac{1}{x}\right)^3 & x \geq 1 \end{cases}$

3. $\mu_x = \frac{\theta}{\theta-1} = \frac{3}{3-1} = \frac{3}{2}$

$$var(x) = \frac{\theta}{(\theta-1)^2\,(\theta-2)} = \frac{3}{(3-1)^2\,(3-2)} = \frac{3}{4}$$

4. $P(x > 2) = 1 - P(x \leq 2) = 1 - F(2) = 1 - 1 + \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

8. Rayleigh Distribution

A random variable X is said to have a Rayleigh distribution with parameter λ if its probability density function (p.d.f.) is given by:

$$f(x,\lambda) = \begin{cases} \frac{x}{\lambda^2} e^{-\frac{x^2}{2\lambda^2}} & x \geq 0 \\ 0 & o.w. \end{cases}$$

Where the parameters $\lambda > 0$.

1. $f(x,\lambda) \geq 0$
2. $\int_{\forall x} f(x,\lambda)dx = 1$
3. The **mean** of the r.v. $X \sim Ray\ (\lambda)$ can be obtain as follows

$$\mu_x = E(x) = \sqrt{\frac{\pi}{2}}\ \lambda$$

4. The **variance** of the r.v. $X \sim Par\ (\theta)$ can be obtain as follows

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{4-\pi}{2}\ \lambda^2$$

5. The Cumulative Distribution Function of X is given by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\ dt = \int_{0}^{x} \frac{t}{\lambda^2} e^{-\frac{t^2}{2\lambda^2}}\ dt$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{x^2}{2\lambda^2}} & x \geq 0 \end{cases}$$

Ex.18: A machine measures vibration, and the vibration follows a Rayleigh distribution with $\lambda = 2$.

1. Write the p.d.f. of X.
2. Find the c.d.f. of X.
3. Find the mean and the variance of X.
4. What is the probability that the vibration is less than 2.

Solution:

1. $f(x,\lambda) = \begin{cases} \frac{x}{4}\ e^{-\frac{x^2}{8}} & x \geq 0 \\ 0 & o.w. \end{cases}$

2. $F(x) = \begin{cases} 0 & x < 0 \\ 1 - \ e^{-\frac{x^2}{8}} & x \geq 0 \end{cases}$

3. $\mu_x = \lambda\sqrt{\frac{\pi}{2}}\ = 2 * \sqrt{\frac{3.14}{2}} = 2 * 1.2533 = 2.50$

$$var(x) = \frac{4-\pi}{2}\ \lambda^2 \ = \frac{4-3.14}{2} * 2^2 = 1.716$$

4. $P(x<2) = F(2) = 1 - \ e^{-\frac{2^2}{8}} = 1 - \ e^{-0.5} = 1 - 0.6065 = 0.3935$