الجامعة المستنصرية
كلية الادارة والاقتصاد
قسم الاحصاء

التوزيعات الاحتمالية

Probability Distributions

المستوى الثاني – الكورس الثاني – مسار بولونيا

2026-2025

استاذة المادة: د. مروة علي

Chapter Four

1. The Mathematical Expectation

The expected value is the average value of a random variable over many repetitions of an experiment. It is a fundamental concept used to summarize the characteristics of a probability distribution.

If X is a discrete random variable with p.m.f. $p(x)$ the expected value of X, $E(x)$ is given by:

$$E(x) = \sum_{\forall x} x\, p(x)$$

If X is a continuous random variable with p.d.f. $f(x)$ the expected value of X, $E(x)$ is given by:

$$E(x) = \int_{\forall x} x\, f(x)\, dx$$

2. The Moments

Moments are expected values of powers of a random variable used to describe the shape and spread of a distribution. The full collection of moments includes the expected values of all positive integral powers of X.

The $r-th$ moment of a random variable X, denoted by $\acute{\mu}_r$, is given by:

1. Moments about the origin

$$\acute{\mu}_r = E(x^r)$$

$\acute{\mu}_1 = E(x^1) = E(x)$ $\quad (First\ Moment)$

$\acute{\mu}_2 = E(x^2)$ $\quad (Second\ Moment)$

$\acute{\mu}_3 = E(x^3)$ $\quad (Third\ Moment)$

⋮

⋮

For a discrete random variable:

$$\acute{\mu}_r = E(x^r) = \sum_{\forall x} x^r \, p(x)$$

For a continuous random variable:

$$\acute{\mu}_r = E(x^r) = \int_{\forall x} x^r \, f(x) \, dx$$

2. Moments about the mean (Central Moments)

$$\mu_r = E(x-\mu)^r$$

$\mu_1 = E(x-\mu)^1 = E(x-\mu) = 0$ $\quad (First \text{ Central } Moment)$

$\mu_2 = E(x-\mu)^2 = variance$ $\quad (Second \text{ Central } Moment)$

$\mu_3 = E(x-\mu)^3$ $\quad (Third \text{ Central } Moment)$

⋮

⋮

For a discrete random variable:

$$\mu_r = E(x-\mu)^r = \sum_{\forall x} (x-\mu)^r \, p(x)$$

For a continuous random variable:

$$\mu_r = E(x-\mu)^r = \int_{\forall x} (x-\mu)^r \, f(x) \, dx$$

Remark: The mean and variance can be calculated using moments. The first moment gives the mean and the second moment helps to find the variance.

$$\mu_x = E(x)$$

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2$$

Theorems:

1. If X be a random variable, $g(x)$ be any function of X then
$$E[g(x)] = \sum_{\forall x} g(x)\,p(x) \qquad if\ X\ is\ discrete\ r.v.$$
$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\,f(x)\,dx \qquad if\ X\ is\ continuous\ r.v.$$
2. If c is constant, then $E(c) = c$
3. $E(cx) = c\,E(x)$
4. $E(x_1 + x_2) = E(x_1) + E(x_2)$ $\quad x_1, x_2$ *are tow random variables*
5. $E(ax_1 + bx_2) = aE(x_1) + bE(x_2)$
6. $var\ (a) = 0$
7. $var\ (ax + b) = a^2\ var(x)$

<u>Ex.1</u>: If X is the value shown when rolling a fair die. Find the expected value of X.

$$p(x) = \begin{cases} \frac{1}{6} & x = 1,2,3,4,5,6 \\ 0 & o.w. \end{cases}$$

Solution:

$$E(x) = \sum_{\forall x} x\,p(x) = \sum_{x=1}^{6} x\,p(x)$$

$$= (1) * p(1) + (2) * p(2) + (3) * p(3) + (4) * p(4) + (5) * p(5) + (6) * p(6)$$

$$= (1) * \frac{1}{6} + (2) * \frac{1}{6} + (3) * \frac{1}{6} + (4) * \frac{1}{6} + (5) * \frac{1}{6} + (6) * \frac{1}{6} = 3.5$$

<u>Ex.2</u>: Let X be the r.v. with the following p.d.f.

$$p(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

Find the mathematical expectation of X.

Solution:

$$E(x) = \int_{\forall x} x\, f(x)\, dx = \int_0^1 x\, 2x\, dx = \int_0^1 2x^2\, dx = 2\left[\frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{3} - \frac{0}{3}\right]$$

$$= \frac{2}{3}$$

<u>Ex.3</u>: Let we have the following p.m.f.

x	-1	0	1	2
$p(x)$	$1/8$	$3/8$	$1/4$	$1/4$

1. Prove that $p(x)$ is a valid probability mass function
2. Compute $E(x), E(x^2), E(2x+1), var(3x+2)$
3. Compute the second central moment (μ_2) of the random variable X.

Solution:

1. $\sum_{\forall x} p(x) = 1 \implies \frac{1}{8} + \frac{3}{8} + \frac{2}{8} + \frac{2}{8} = \frac{8}{8} = 1$

2. $E(x) = \sum_{\forall x} x\, p(x) = \sum_{x=-1}^{2} x\, p(x)$

$$= (-1) * \frac{1}{8} + (0) * \frac{3}{8} + (1) * \frac{1}{4} + (2) * \frac{1}{4} = \frac{5}{8} = 0.625$$

$$E(x^2) = \sum_{\forall x} x^2\, p(x) = \sum_{x=-1}^{2} x^2\, p(x)$$

$$= (-1)^2 * \frac{1}{8} + (0)^2 * \frac{3}{8} + (1)^2 * \frac{1}{4} + (2)^2 * \frac{1}{4} = \frac{11}{8}$$

$$E(2x+1) = 2E(x) + E(1) = 2 * \frac{5}{8} + 1 = \frac{18}{8}$$

$$var(x) = E(x^2) - (E(x))^2 = \frac{11}{8} - \left(\frac{5}{8}\right)^2 = \frac{11}{8} - \frac{25}{64} = \frac{63}{64}$$

$$var(3x+2) = 9\, var(x) + 0 = 9\, var(x) = 9 * \frac{63}{64} = \frac{567}{64}$$

3. $\mu_2 = E(x-\mu)^2 = \sum_{x=-1}^{2} (x-\mu)^2\, p(x)$

$$= (-1-0.625)^2 * \frac{1}{8} + (0-0.625)^2 * \frac{3}{8} + (1-0.625)^2 * \frac{1}{4} + (2-0.625)^2 * \frac{1}{4} = 0.9845$$

<u>Ex.4</u>: Let we have the following p.d.f.

$$f(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

1. Prove that $f(x)$ is a valid probability mass function
2. Compute $E(x), E(x^2), E(5x), var(6x)$.

Solution:

1. $\int_{\forall x} f(x)dx = 1$

$$6\int_0^1 (x-x^2)dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 6\left[\frac{1}{2} - \frac{1}{3}\right] = 3 - 2 = 1$$

2. $E(x) = \int_{\forall x} x\, f(x)\, dx = 6\int_0^1 x\,(x-x^2)\, dx = 6\int_0^1 \,(x^2 - x^3)\, dx$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3} - \frac{1}{4}\right] = 6\left[\frac{1}{12}\right] = \frac{1}{2}$$

$$E(x^2) = \int_0^1 x^2\, f(x)\, dx = \int_0^1 x^2\, 6x(1-x)\, dx = 6\int_0^1 \,(x^3 - x^4)\, dx$$

$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{5}\right] = 6\left[\frac{1}{20}\right] = \frac{6}{20}$$

$$E(5x) = 5E(x) = 5 * \frac{1}{2} = \frac{5}{2}$$

$$var(x) = E(x^2) - (E(x))^2 = \frac{6}{20} - \left(\frac{1}{2}\right)^2 = \frac{6}{20} - \frac{1}{4} = \frac{6}{20} - \frac{5}{20} = \frac{1}{20}$$

$$var(6x) = 36 * var(x) = 36 * \frac{1}{20} = \frac{36}{20}$$

Ex.5: Find the mean and variance of a discrete uniform distribution using its moments.

Solution:

$$p(x,n)=\begin{cases}1/n & x=1,2,3,\dots,n\\ 0 & otherwise\end{cases}$$

$1.\mu_x=E(x)=\sum_{\forall x} x\, p(x)=\sum_{x=1}^{n} x\,\frac{1}{n}=\frac{1}{n}\sum_{x=1}^{n} x$

$$=\frac{1}{n}\,(1+2+\cdots+n)$$

Recall that $\sum_{i=1}^{n} i\ =\frac{n(n+1)}{2}$

$$\mu_x=\frac{1}{n}\,(1+2+\cdots+n)=\frac{1}{n}*\frac{n(n+1)}{2}=\frac{(n+1)}{2}$$

2. $var(x)=\sigma_x^2=E(x^2)-(E(x))^2$

$$E(x^2)=\sum_{\forall x} x^2\, p(x)=\sum_{x=1}^{n} x^2\,\frac{1}{n}=\frac{1}{n}\sum_{x=1}^{n} x^2$$

$$=\frac{1}{n}\,(1^2+2^2+\cdots+n^2)$$

Recall that $\sum_{i=1}^{n} i^2\ =\frac{n(n+1)(2n+1)}{6}$

$$E(x^2)=\frac{1}{n}\,(1^2+2^2+\cdots+n^2)=\frac{1}{n}*\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{(n+1)(2n+1)}{6}$$

$$var(x)=\sigma_x^2=E(x^2)-(E(x))^2$$

$$var(x)=\frac{(n+1)(2n+1)}{6}-\left(\frac{n+1}{2}\right)^2=\frac{(n+1)(2n+1)}{6}-\frac{(n+1)^2}{4}$$

$$=\frac{2(n+1)(2n+1)-3(n+1)^2}{12}$$

$$=\frac{2[2n^2+n+2n+1]-3[n^2+2n+1]}{12}$$

$$var(x) = \sigma_x^2 = \frac{4n^2 + 6n + 2 - 3n^2 - 6n - 3}{12} = \frac{n^2 - 1}{12}$$

<u>Ex.6</u>: Find the mean and variance of a Burnoulli distribution using its moments.

Solution:

$$p(x,p) = \begin{cases} p^x\,(1-p)^{1-x} & x = 0,1 \\ 0 & otherwise \end{cases}$$

1.$\mu_x = E(x) = \sum_{\forall x} x\, p(x) = \sum_{x=0}^{1} x\; p^x\;(1-p)^{1-x}$

$$= (0)p^0(1-p)^{1-0} + (1)p^1(1-p)^{1-1} = p$$

2. $var(x) = \sigma_x^2 = E(x^2) - (E(x))^2$

$$E(x^2) = \sum_{\forall x} x^2\; p(x) = \sum_{x=0}^{1} x^2\; p^x\;(1-p)^{1-x}$$

$$= (0)^2p^0(1-p)^{1-0} + (1)^2p^1(1-p)^{1-1} = p$$

$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2$

$$= p - p^2 = p(1-p) = pq$$

<u>Ex.7</u>: Find the mean and variance of a continuous uniform distribution using its moments.

Solution:

$$f(x,a,b) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & o.w. \end{cases}$$

1.$\mu_x = E(x) = \int_{\forall x} x\, f(x)\, dx = \int_a^b x\, \frac{1}{b-a}\, dx = \frac{1}{b-a}\int_a^b x\;\; dx$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a} * \frac{b^2-a^2}{2} = \frac{1}{b-a} * \frac{(b-a)(b+a)}{2} = \frac{a+b}{2}$$

2. $var(x)=\sigma_x^2 = E(x^2)-(E(x))^2$

$$E(x^2) = \int_{\forall x} x^2\, f(x)\, dx = \int_a^b x^2 \,\frac{1}{b-a}\, dx = \frac{1}{b-a}\int_a^b x^2\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a} * \frac{b^3-a^3}{3} = \frac{1}{b-a} * \frac{(b-a)(b^2+ab+a^2)}{3}$$

$$= \frac{a^2+ab+b^2}{3}$$

$var(x)=\sigma_x^2 = E(x^2)-(E(x))^2$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4(a^2+ab+b^2)-3(a^2+2ab+b^2)}{12}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12} = \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}$$

Ex.8: Find the mean and variance of a exponential distribution using its moments.

Solution:

$$f(x,\theta) = \begin{cases} \theta\, e^{-\theta x} & x>0 \\ 0 & o.w. \end{cases}$$

1.$\mu_x = E(x) = \int_{\forall x} x\, f(x)\, dx = \int_0^\infty x\,\theta\, e^{-\theta x}\, dx = \theta\int_0^\infty x\;\; e^{-\theta x}\, dx$

But $\int_0^\infty x^{\alpha-1}\;\; e^{-\beta x}\, dx = \frac{\Gamma\alpha}{\beta^\alpha} \quad ,\alpha=2\quad \beta=\theta$

$\mu_x = E(x) = \theta * \frac{\Gamma 2}{\theta^2} = \frac{1}{\theta}$

2. $var(x)=\sigma_x^2 = E(x^2)-(E(x))^2$

$$E(x^2) = \int_{\forall x} x^2 \, f(x)\, dx = \int_0^\infty x^2 \, \theta \, e^{-\theta x} \, dx = \theta \int_0^\infty x^2 \; e^{-\theta x} \, dx$$

But $\int_0^\infty x^{\alpha-1} \; e^{-\beta x} \, dx = \frac{\Gamma\alpha}{\beta^\alpha} \quad , \alpha = 3 \quad \beta = \theta$

$E(x^2) = \theta * \frac{\Gamma 3}{\theta^3} = \frac{2}{\theta^2}$

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2 = \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

<u>Ex.9</u>: Let $X \sim exp(2)$. Find

$1.\mu_x \quad 2.\sigma_x^2 \quad 3.E(3x) \quad 4.var(2x+1)$

Solution:

$$f(x,2) = \begin{cases} 2\, e^{-2x} & x > 0 \\ 0 & o.w. \end{cases}$$

$1.\ \mu_x = E(x) = \frac{1}{\theta} = \frac{1}{2}$, $\qquad 2.var(x) = \sigma_x^2 = \frac{1}{\theta^2} = \frac{1}{4}$

$3.\ E(3x) = 3E(x) = 3 * \frac{1}{2} = \frac{3}{2}$

$$4.var(2x+1) = 4\ var(x) + 0 = 4 * \frac{1}{4} = 1$$

<u>Ex.10</u>: Let $X \sim Po(3)$. Find

$1.\mu_x \quad 2.\sigma_x^2 \quad 3.E(2x+1) \quad 4.var(3x+9)$

Solution:

$$p(x,\lambda) = \begin{cases} \frac{e^{-3}\, 3^x}{x!} & x = 0,1, \dots\dots \\ 0 & otherwise \end{cases}$$

$1.\ \mu_x = E(x) = \lambda = 3 \qquad 2.var(x) = \sigma_x^2 = \lambda = 3$

$3.E(2x+1) = 2E(x) + 1 = 2 * 3 + 1 = 7$

$4.var(3x+9) = 9\ var(x) + 0 = 9 * 3 = 27$

3. The Moment Generating Function

The moment generating function denoted by $M_x(\text{t})$ for a random variable X is defined to be the expectation of the exponential function e^{tx}. Where **t** all real numbers.

$$M_x(\text{t}) = E(e^{tx}) = \sum_{\forall x} e^{tx}\, p(x) \qquad if\ X \text{ is discrete random variable}$$

$$M_x(\text{t}) = E(e^{tx}) = \int_{\forall x} e^{tx}\, f(x)dx \qquad if\ X \text{ is continuous random variable}$$

We call $M_x(\text{t})$ the moment generating function because all of the moments of X can be obtained by differentiating $M_x(\text{t})$ and then evaluating the result at t=0.

$$\acute{M}_x(0) = E(x) = m_1 \qquad 1st\ moment$$

$$\acute{\acute{M}}_x(0) = E(x^2) = m_2 \qquad 2nd\ moment$$

$$\acute{\acute{\acute{M}}}_x(0) = E(x^3) = m_3 \qquad 3rd\ moment$$

$$\vdots$$

$$M_x^r(0) = E(x^r) = m_r \qquad rth\ moment$$

Ex.11: A fair coin is flipped twice, let X be the number of heads that occur.

1. Find the MGF.
2. Find the mean and variance of X using the MGF.

Solution:

$$p(x) = \begin{cases} 1/4 & x = 0 \\ 1/2 & x = 1 \\ 1/4 & x = 2 \\ 0 & o.w. \end{cases}$$

1. $M_x(\text{t}) = E(e^{tx}) = \sum_{\forall x} e^{tx}\, p(x) = \sum_{x=0}^{2} e^{tx}\, p(x)$

$$= e^{t(0)}\, p(0) + e^{t(1)}\, p(1) + e^{t(2)}\, p(2)$$

$$= e^0 * \frac{1}{4} + e^t * \frac{1}{2} + e^{2t} * \frac{1}{4}$$

$$M_x(\text{t}) = \frac{1}{4} + \frac{1}{2}e^t + \frac{1}{4}e^{2t}$$

2. $\acute{M}_x(0) = E(x) = \mu_x$

$$\acute{M}_x(\text{t}) = 0 + \frac{1}{2}e^t + \frac{1}{4}e^{2t}(2) = \frac{1}{2}e^t + \frac{1}{2}e^{2t}$$

$$E(x) = \mu_x = \acute{M}_x(0) = \frac{1}{2}e^0 + \frac{1}{2}e^0 = \frac{1}{2} + \frac{1}{2} = 1$$

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2$$

$$\acute{\acute{M}}_x(\text{t}) = \frac{1}{2}e^t + \frac{1}{2}e^{2t}(2) = \frac{1}{2}e^t + e^{2t}$$

$$E(x^2) = \acute{\acute{M}}_x(0) = \frac{1}{2}e^0 + e^0 = \frac{1}{2} + 1 = \frac{3}{2}$$

$$var(x) = E(x^2) - (E(x))^2 = \frac{3}{2} - (1)^2 = \frac{1}{2}$$

<u>Ex.12</u>: Let X be a random variable with the following p.d.f.

$$f(x) = \begin{cases} 2\, e^{-2x} & x > 0 \\ 0 & o.w. \end{cases}$$

1. Find the m.g.f. of X.
2. Use the m.g.f. to find the mean and variance.
3. Use the m.g.f. to find 3rd moment of X.

Solution:

1. $M_x(\text{t}) = E(e^{tx}) = \int_{\forall x} e^{tx} f(x)dx$

$$M_x(\text{t}) = \int_0^\infty e^{tx}\, 2\, e^{-2x}dx = 2\int_0^\infty e^{tx}\ e^{-2x}dx = 2\int_0^\infty e^{-2x+tx}\ dx$$

$$= 2\int_0^\infty e^{-(2-t)x}\ dx = \frac{-2}{(2-t)}\int_0^\infty -(2-t)e^{-(2-t)x}\ dx$$

$$M_x(\mathrm{t}) = \frac{-2}{(2-t)}\left[e^{-(2-t)x}\right]_0^{\infty} = \frac{-2}{(2-t)}\left[e^{-\infty} - e^0\right] = \frac{2}{(2-t)} = 2(2-t)^{-1}$$

2. $\acute{M}_x(0) = E(x) = \mu_x$

$$\acute{M}_x(\mathrm{t}) = 2(-1)(2-t)^{-2}(-1) = 2(2-t)^{-2}$$

$$\acute{M}_x(0) = 2(2-0)^{-2} = \frac{2}{4} = \frac{1}{2}$$

$$\acute{\acute{M}}_x(\mathrm{t}) = 2(-2)(2-t)^{-3}(-1) = 4(2-t)^{-3}$$

$$E(x^2) = \acute{\acute{M}}_x(0) = 4(2-0)^{-3} = \frac{4}{8} = \frac{1}{2}$$

$$var(x) = E(x^2) - (E(x))^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

3. $\acute{\acute{\acute{M}}}_x(\mathrm{t}) = 4(-3)(2-t)^{-4}(-1) = 12(2-t)^{-4}$

$$\acute{\acute{\acute{M}}}_x(0) = E(x^3) = 12(2-0)^{-4} = \frac{12}{16} = \frac{3}{4}$$

<u>Ex.13</u>: Let X be a Bernoulli random variable with parameter p

1. Find the m.g.f. of X.
2. Use the m.g.f. to find the mean and variance.

Solution:

$$p(x,p) = \begin{cases} p^x\,(1-p)^{1-x} & x = 0,1 \\ 0 & otherwise \end{cases}$$

1. $M_x(\mathrm{t}) = E(e^{tx}) = \sum_{\forall x} e^{tx}\, p(x) = \sum_{x=0}^{1} e^{tx}\, p^x\,(1-p)^{1-x}$

$$= e^{t(0)}\, p^{(0)}\,(1-p)^{1-0} + e^{t(1)}\, p^{(1)}\,(1-p)^{1-1} = q + pe^t$$

2. $\acute{M}_x(0) = E(x) = \mu_x$

$$\acute{M}_x(\mathrm{t}) = 0 + pe^t = pe^t$$

$$E(x) = \mu_x = \acute{M}_x(0) = pe^0 = p$$

$$\acute{\acute{M}}_x(\mathrm{t}) = pe^t$$

$$E(x^2) = \acute{\acute{M}}_x(0) = pe^0 = p$$

$$var(x) = E(x^2) - (E(x))^2 = p - (p)^2 = p(1-p) = pq$$

Ex.14: Let X be a continuous uniform distribution with parameters (a, b). Find the m.g.f. of X.

Solution:

$$f(x, a, b) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & o.w. \end{cases}$$

$$M_x(\mathrm{t}) = E(e^{tx}) = \int_{\forall x} e^{tx} f(x) dx = \int_a^b e^{tx} \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx} \; dx = \frac{1}{b-a} * \frac{1}{t} \int_a^b t\, e^{tx} \; dx = \frac{1}{b-a} * \frac{1}{t} \; [e^{tx}]_a^b$$

$$M_x(\mathrm{t}) = \frac{e^{tb} - e^{ta}}{(b-a)t}$$

Ex.15: Let X be a negative binomial distribution with parameters (r, p). Find the m.g.f. of X.

Solution:

$$p(x, r, p) = \begin{cases} C_x^{x+r-1} \; p^r \; q^x & x = 0,1,2, \dots \dots \\ 0 & otherwise \end{cases}$$

$$M_x(\mathrm{t}) = E(e^{tx}) = \sum_{\forall x} e^{tx} \, p(x) = \sum_{x=0}^{\infty} e^{tx} \, C_x^{x+r-1} \; p^r \; q^x$$

$$= p^r \sum_{x=0}^{\infty} e^{tx} \, C_x^{x+r-1} \; q^x = p^r \sum_{x=0}^{\infty} C_x^{x+r-1} \; (e^t q)^x$$

Recall that $\sum_{j=0}^{\infty} C_j^{j+r-1} \; x^j = (1-x)^{-r}$

$$M_x(\mathrm{t}) = p^r (1 - e^t q)^{-r} = \left(\frac{p}{1 - qe^t}\right)^r$$

4. Law of Large Number

This theorem deals with the fact that under some conditions the average of a sequence of random variables converges to the expected average.

Let $X_1, X_2, \dots \dots, X_n$ are a random sample of size n drawn from a population with a finite mean μ. Let $\bar{X}_n$ denote the sample mean. Then, for any $\epsilon > 0$

$$P\{|\bar{X}_n - \mu| > \epsilon\} \longrightarrow 0 \; as \; n \longrightarrow 0$$

5. Central Limit Theorem

Suppose $X_1, X_2, \dots \dots, X_n$ are a random sample of size n drawn from a population with a finite mean μ and a finite variance σ^2. Let $\bar{X}_n$ denote the sample mean. Consider the normalized random variable:

$$Z_n = \frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{var\,(\bar{X}_n)}}$$

Then as $n \longrightarrow \infty$, the distribution of Z converges to the standard normal distribution $Z \sim SN(0,1)$.

<u>Ex.16</u>: Let X be the number of successes in 7000 Bernoulli trials with probability of success of 0.7 on a given trial use the central limit theorem to estimate:

1. $P(x < 4950)$
2. $P(4750 < x < 5000)$

Solution:

$X \sim Bin(7000, 0.7)$

$\mu = n * p = 7000 * 0.7 = 4900$

$\sigma^2 = n * p * q = 7000 * 0.7 * 0.3 = 1470$

$\sigma = \sqrt{1050} = 38.34$

$X \sim N(4900, 1470)$

1. $P(x < 4950) = P\left(\frac{x-\mu}{\sigma} < \frac{4950-4900}{38.34}\right) = P(Z < 1.30)$

$$= \phi(1.30) = 0.9032$$

2. $P(4750 < x < 5000) = P\left(\frac{4750-4900}{38.34} < \frac{x-\mu}{\sigma} < \frac{5000-4900}{38.34}\right)$

$$= P(-3.91 < Z < 2.61) = \phi(2.61) - \phi(-3.91)$$

$$= \phi(2.61) - [1 - \phi(3.91)] = 0.99547 - [1 - 0.99995] = 0.995$$

Ex.17: Let $\bar{X}_{15}$ denote the mean of a random sample of size 15 drawn from a distribution whose p.d.f. is:

$$f(x) = \begin{cases} \frac{3}{2}x^2 & -1 < x < 1 \\ 0 & o.w. \end{cases} \quad , \text{ and } \mu = 0, \sigma^2 = \frac{3}{5}$$

Use the central limit theorem to compute $P(0.03 \leq \bar{X} \leq 0.15)$.

Solution:

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$E(\bar{X}_n) = E(\bar{X}_{15}) = \mu = 0$$

$$var(\bar{X}_n) = var(\bar{X}_{15}) = \frac{\sigma^2}{n} = \frac{3/2}{15} = \frac{3}{75}$$

$$P(0.03 \leq \bar{X} \leq 0.15) = P\left(\frac{0.03 - 0}{\sqrt{3/75}} \leq Z \leq \frac{0.15 - 0}{\sqrt{3/75}}\right)$$

$$= P(0.15 < Z < 0.75) = \phi(0.75) - \phi(0.15)$$
$$= 0.77337 - 0.55982 = 0.2135$$

Ex.18: If 10 fair dice are rolled, find the approximate probability that the sum obtained is between 30 and 40. Use the central limit theorem. Assume that for each die, $\mu_x = 3.5, \sigma_x^2 = \frac{35}{12}$.

Solution:

$P\left(30 \leq \sum_{i=1}^{10} x_i \leq 40\right) \qquad Let\ S = \sum_{i=1}^{10} x_i$

$$\mu_s = E(s) = E\sum_{i=1}^{10} x_i = \sum_{i=1}^{10} E(x_i) = \sum_{i=1}^{10} 3.5 = 10 * 3.5 = 35$$

$$\sigma_s^2 = var(s) = var\sum_{i=1}^{10} x_i = \sum_{i=1}^{10} var(x_i) = \sum_{i=1}^{10} \frac{35}{12} = 10 * \frac{35}{12} = \frac{350}{12}$$

$$P(30 \leq S \leq 40) = P\left(\frac{30-35}{\sqrt{350/12}} \leq \frac{s-\mu_s}{\sqrt{\sigma_s^2}} \leq \frac{40-35}{\sqrt{350/12}}\right)$$

$$= P(-0.93 \leq Z \leq 0.93) = \phi(0.93) - \phi(-0.93)$$

$$= \phi(0.93) - [1 - \phi(0.93)] = 0.8238 - [1 - 0.8238] = 0.65$$

Ex.19: Let X random variable has the following p.d.f.

$$f(x) = \begin{cases} \frac{x^3}{4} & 0 < x < 2 \\ 0 & o.w. \end{cases}$$

Let $\bar{X}_{25}$ denote the mean of a random sample of size 25. Use the central limit theorem to compute $P(1.5 \leq \bar{X} \leq 1.65)$.

Solution:

$$P(1.5 \leq \bar{X} \leq 1.65) = P\left(\frac{1.5 - E(\bar{X}_n)}{\sqrt{var\ (\bar{X}_n)}} \leq \frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{var\ (\bar{X}_n)}} \leq \frac{1.65 - E(\bar{X}_n)}{\sqrt{var\ (\bar{X}_n)}}\right)$$

$$E(x) = \int_{\forall x} x\, f(x)dx = \int_0^2 x\, \frac{x^3}{4} dx = \int_0^2 \frac{x^4}{4} dx = \frac{1}{4}\left[\frac{x^5}{5}\right]_0^2 = \frac{8}{5} = 1.6$$

$$E(\bar{X}_n) = E(\bar{X}_{25}) = \mu = 1.6$$

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2$$

$$E(x^2) = \int_{\forall x} x^2\, f(x)dx = \int_0^2 x^2\, \frac{x^3}{4} dx = \int_0^2 \frac{x^5}{4} dx = \frac{1}{4}\left[\frac{x^6}{6}\right]_0^2 = \frac{8}{3}$$

$$var(x) = \sigma_x^2 = \frac{8}{3} - (1.6)^2 = \frac{8}{75} = 0.106$$

$$var\,(\bar{X}_n) = var\,(\bar{X}_{25}) = \frac{\sigma_x^2}{n} = \frac{0.106}{25} = 0.004$$

$$\bar{X} \sim N(1.6\,, 0.004)$$

$$P(1.5 \leq \bar{X} \leq 1.65) = P\left(\frac{1.5 - 1.6}{\sqrt{0.004}} \leq \frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{var\,(\bar{X}_n)}} \leq \frac{1.65 - 1.6}{\sqrt{0.004}}\right)$$

$$P(1.5 \leq \bar{X} \leq 1.65) = P(-1.53 \leq Z_n \leq 0.77) = \phi(0.77) - \phi(-1.53)$$

$$= \phi(0.77) - [1 - \phi(1.53)] = 0.7793 - [1 - 0.9369]$$

$$= 0.7162$$